Effective Pedagogy in Mathematics/Pāngarau

Best Evidence Synthesis Iteration [BES]

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This report is one of a series of best evidence synthesis iterations (BESs) commissioned by the Ministry of Education. The Iterative Best Evidence Synthesis Programme is seeking to support collaborative knowledge building and use across policy, research and practice in education. BES draws together bodies of research evidence to explain what works and why to improve education outcomes, and to make a bigger difference for the education of all our children and young people.

Each BES is part of an iterative process that anticipates future research and development informing educational practice. This BES follows on from other BESs focused on quality teaching for diverse learners in early childhood education and schools. Its use will be informed by other BESs, focused on teacher professional learning and development and educational leadership. These documents will progressively become available at: [http://educationcounts.edcentre.govt.nz/goto/BES](http://educationcounts.edcentre.govt.nz/goto/BES)

Feedback is welcome at best.evidence@minedu.govt.nz

Note: the references printed in purple refer to a list of URLs in Appendix 2. These are a selection of potentially useful sources for teachers to engage more deeply with the range of issues raised in this best evidence synthesis iteration.
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About the writers

Glenda Anthony and Margaret Walshaw, both from the School of Curriculum and Pedagogy at Massey University, bring to this Best Evidence Synthesis (BES) decades of mathematics classroom teaching and educational research experience. They are acutely aware of the challenge that educators face in constructing a democratic mathematical community with which all students can identify. For them, making a positive difference to diverse learners’ outcomes is a central educational issue. At the heart of their work is a concerted effort to illuminate how this issue is best addressed. In this synthesis, they report on the outcome of their deliberations over, and search for, what makes a difference for diverse learners in mathematics/pāngarau.

Advisory Group

A core Advisory Group membership was selected to provide expertise and critique in relation to the various focuses of the BES, including Māori and Pasifika learners, early childhood, primary and secondary sectors, and teacher education. The authors wish to thank the members of this group:

- Dr Ian Christensen (Massey University and He Kupenga Hao i te Reo)
- Dr Joanna Higgins (Victoria University of Wellington)
- Roberta Hunter (Massey University)
- Garry Nathan (Auckland University)
- Dr Sally Peters (Waikato University)
- Assoc. Prof. Jenny Young-Loveridge (Waikato University)

We also wish to acknowledge the supportive formative feedback received from Faith Martin (Director, Massey Child Care Centre), Brian Paewai (Runanga Kura Kaupapa Māori), Professor Anne Smith (University of Otago) and Johanna Wood (Principal, Queen Elizabeth College, Palmerston North).

Ministry of Education advisory team

The Ministry of Education, led by Dr Adrienne Alton-Lee, has guided the development of the synthesis. The team at the Ministry also gave us access to additional literature and demographic and trend data. We thank all of the team.

External quality assurance

Professor Paul Cobb from Vanderbilt University, US, has provided invaluable assistance. We would like to acknowledge his scholarly critique and thank him for his knowledgeable contribution to the synthesis.

Formative quality assurance was also provided by: Maggie Haynes (Unitec), Professor Derek Holton (University of Otago), Tamsin Meaney (EARU, University of Otago), Lynne Peterson, Tony Trinick (Auckland University), initial and ongoing Teacher Education (Victoria University of Wellington), the New Zealand Educational Institute and representation from the Post Primary Teachers’ Association (Jill Gray). We wish to thank them all for their contributions.
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The Ministry of Education extends special thanks to those who have contributed to different stages of this BES development through their participation in the BES Management Group. The advice and guidance from principal Diane Leggett, NZEI, and Judie Alison, Advisory Officer (Professional Issues), PPTA, have greatly strengthened this BES development. Particular thanks also to Robina Broughton and Linda Gendall, New Zealand Teachers Council, and to Ministry of Education colleagues.

The Chief Education Adviser acknowledges in particular the support and guidance provided by Malcolm Hyland and Ro Parsons through the partnership between BES and the Numeracy Development Project. The model of collaboration across research, practice and policy exemplified in that project has been an inspiration for the Iterative Best Evidence Synthesis Programme. Thanks to all those in the wider NDP community who have informed the BES development.

The Ministry of Education thanks Dr Fred Biddulph for his ongoing role in providing advice from the earliest formulation of the request for proposals through to a consideration of the final draft.

The Ministry of Education is indebted to Professor Bill Barton, Mathematics Education Unit, University of Auckland, for taking a proactive leadership role in bringing together teacher, teacher educator and research colleagues from across New Zealand to assist in scoping this BES at the outset.

Thanks for the deeply valued contribution made to the formative quality assurance and other advice offered to this BES development by Professor Paul Cobb, Vanderbilt University; Irene Cooper, Sandie Aikin, Cheryl Baillie and colleagues, NZEI, Jill Gray and Patrick McEntee, PPTA; Dr Mere Skerrett-White, Dr Maggie Haynes, Unitech, Dr Jo Higgins and colleagues at the Victoria University of Wellington College of Education; Dr Tamsin Meany, Professor Derek Holton, Lynne Petersen, Peter Hughes, Lynn Tozer and Michael Drake.

Thanks also for the valued participation of colleagues from initial teacher education institutions across New Zealand and Derek Glover, Secretary, New Zealand Association of Mathematics Teachers, in the formative quality assurance forum held for this BES development.

Thanks also for the significant contribution made to this and other BES developments through the advice given in the development of the Guidelines for Generating a Best Evidence Synthesis Iteration by the BES Standards Reference Group; The BES Māori Educational Research Advisory Group, the BES Pasifika Educational Research Advisory Group and Associate Professor Brian Haig, University of Canterbury.
Forewords

International

Even the casual visitor is struck by the dramatic changes that have occurred in New Zealand in the last 15 years. I have tuned in to local media on each of my four visits to get an initial sense of people's current concerns and issues. Based on this narrow sampling, the New Zealand of 1991 was an immensely likeable country that had seen better days and was struggling to find its place in a rapidly changing world. Although innovation and experimentation appeared to be the watchwords of the day, there seemed to be an undercurrent of apprehension and anxiety as people attempted to cope with economic disruption. Today, New Zealand continues to be an immensely likeable place, but the visitor immediately notices a quiet, understated self-assurance. It has become a largely prosperous country that, in a very real sense, has reinvented itself as a leading information economy in an increasingly globalised world. Refreshingly for the visitor from the United States, there appears to be widespread belief that government will approach problems pragmatically and is capable of solving them. If the Iterative Best Evidence Synthesis Programme is representative of New Zealand government in action, this belief would appear to be well founded.

Put quite simply, the Iterative BES Programme is the most ambitious effort I have encountered that uses rigorous scientific evidence to guide the ongoing improvement of an education system at a national level. The programme has a strong pragmatic bent and is clearly grounded in the hard-won experience of synthesising research findings to inform both policy and teachers' instructional practices. Four aspects of the programme are particularly noteworthy. The first is the overriding commitment to make the development of the best evidence syntheses transparent. This commitment takes concrete form in the exacting evaluation and feedback process that all BES reports undergo at each phase of their development, from the initial identification of relevant bodies of research literature through to the final critique and revision of the report. This is in the best traditions of science, where claims are justified in terms of the means by which they have been produced.

The second notable characteristic is a mature view of evidence and an emphasis on methodological and theoretical pluralism. This is important, given that attempts have been made in a number of countries, including the United States, to legislate what counts as scientific research in education on the basis of ideological adherence to a particular methodology. In taking an inclusive approach, the Iterative BES Programme acknowledges that different types of knowledge are of greatest use to teachers and to policymakers. Teachers make pedagogical decisions on the basis of a detailed understanding of specific students in particular classrooms at particular points in time. Policymakers, in contrast, typically need knowledge of trends and patterns that hold up across classrooms to make decisions that affect large numbers of students and teachers in multiple schools. Different methodologies are appropriate for developing these equally important types of knowledge.

The third noteworthy characteristic of the programme is its focus on the explanatory power and coherence of theories. Priority is given to theories that give insight into learning processes and the specific means of supporting their realisation in classrooms. This pragmatic criterion is important in a field where theoretical perspectives continue to proliferate.

The final notable characteristic of the programme is its explicit attention to the issues of language and culture. This emphasis is clearly critical if New Zealand teachers and policymakers are to address the inequities inherent in the disturbingly large gaps in school achievement between children of different ethnic and racial groups. In keeping with the tenet of methodological and theoretical pluralism, the Iterative BES Programme uses group categories such as socioeconomic status, ethnicity, and culture as key variables in assessing efforts to achieve
equity. However, it avoids stereotyping children of particular racial, ethnic, or language
groups by acknowledging the complexity of individual identity when explaining inequities in
children’s learning opportunities. Furthermore, the programme emphasises ecological models
of learning that link what is happening in classrooms both to the institutional contexts in which
classrooms are located and to issues of race, culture, and language. It is here that the full
ambition of the programme becomes apparent: few viable models of this type currently exist
in education. The BES writers are therefore charged with the task of synthesising in the true
sense of the term, that is, to combine disparate and sometimes fragmented bodies of research
into a single, unified whole. At the risk of understatement, this is a formidable challenge.

The writers of this Best Evidence Synthesis of Effective Pedagogy in Mathematics/Pāngarau,
Drs. Glenda Anthony and Margaret Walshaw, have risen to the challenge. They were charged
with the daunting task of reviewing, organising, and synthesising all mathematics education
research from the early childhood years through secondary school that relates classroom
processes to student learning. On my reading, the resulting synthesis of over 600 research
studies is directly relevant to teachers and will be educative for policymakers. The educative
value of the report stems from Anthony and Walshaw’s focus on what goes on in mathematics
classrooms, thereby providing a window on the complexity of effective pedagogy. The forms
of pedagogical practice that they identify as effective are ambitious because they involve high
expectations for all children’s mathematical learning. The goals at which these forms of
pedagogy aim are best illustrated in chapter 7, A Fraction of the Answer, in which Anthony and
Walshaw pull together the key insights of the proceeding chapters as they present an integrated
series of cases that focus on the learning and teaching of fractions. As this chapter makes
clear, the instructional goals for fractions are not limited to ensuring that children can add,
subtract, multiply, and divide fractions successfully. Instead, the instructional objectives also
focus on children’s development of a deep understanding of fractions as amounts or quantities.
At an elementary level, children who are coming to understand fractions as quantities know
that ¼ is smaller than ½ because there will be more pieces when something is divided into 6
pieces than into 5 pieces, so the pieces must be smaller. At a more advanced level, students
will be able to describe real world situations that involve multiplying and dividing fractional
quantities. More generally, ambitious pedagogy focuses on central mathematical ideas and
principles that give meaning to computational methods and strategies.

Anthony and Walshaw’s review of the relevant research indicates that central mathematical
ideas and principles cannot be directly transmitted to children. However, the research
also shows that discovery approaches that place children in rich environments and simply
encourage them to inquire are also ineffective. Effective pedagogy is complex because it
requires teachers to achieve a significant mathematical agenda by taking children’s current
knowledge and interests as the starting point. As Anthony and Walshaw clarify, these forms
of pedagogy involve a distinctive orientation towards teaching. First and foremost, the
emphasis is on building on students’ existing proficiencies rather than filling gaps in students’
knowledge and remediating weaknesses. As a consequence, the teacher’s focus when planning
for instruction is not on students’ limitations but on their current mathematical competencies
and interests, as these constitute resources on which the teacher can build. More generally,
effective mathematical pedagogy places students’ reasoning at the center of instructional
decision making. As a consequence, the ongoing assessment of students’ reasoning is an
integral aspect of instruction, not a separate activity conducted after the fact to check whether
goals for students’ learning have been achieved. A key characteristic of accomplished teachers
is that they continually adjust instruction, as informed by these ongoing assessments.

One of the strengths of Anthony and Walshaw’s synthesis is that it provides the reader with
a concrete image of what effective mathematical pedagogy looks like. Anthony and Walshaw
emphasise that a respectful, non-threatening classroom atmosphere in which all students feel
comfortable in making contributions is necessary but not, by itself, sufficient. As they document,
the research findings indicate unequivocally that it is also essential that classroom activity
and discourse focus explicitly on central mathematical ideas and processes. The selection of instructional tasks is therefore critical. On the one hand, it is important that task contexts or scenarios are accessible to all students, regardless of cultural background. On the other, the teacher should be able to capitalise on students’ solutions to support their development of increasingly sophisticated forms of mathematical reasoning. Thus, when designing and selecting tasks, the teacher has to take account both of students’ current competencies and interests and their long-term learning goals. As Anthony and Walshaw discuss in chapter 5, an important way in which the teacher can build students’ solutions is by introducing judiciously chosen tools and representations. A second, equally important way in which the teacher can capitalise on the potential of worthwhile mathematical tasks is to engage students in justification, abstraction, and generalisation (see chapter 4), by doing which they learn to speak the language of mathematics.

The image of effective mathematical pedagogy that emerges from Anthony and Walshaw’s synthesis is of teaching as a coherent system rather than a set of discrete, interchangeable strategies. This pedagogical system encompasses:

• a non-threatening classroom atmosphere;
• instructional tasks;
• tools and representations;
• classroom discourse.

To see that these four aspects of effective pedagogy constitute a system, note that the way in which instructional tasks are realised in the classroom and experienced by students depends on the classroom atmosphere, the tools and representations available for them to use, and the nature and focus of classroom discourse. And because effective pedagogy is a system, it makes little sense to think of student learning as being caused by isolated teacher actions or strategies. It is for this reason that Anthony and Walsh speak of mathematical learning being occasioned by teaching. In using this term, Anthony and Walshaw emphasise the teacher’s proactive role in supporting students’ development of increasingly sophisticated forms of mathematical reasoning.

In addition to highlighting the systemic character of effective mathematical pedagogy, Anthony and Walshaw make good on the charge to develop an ecological model of learning that links what is happening in the classroom to issues of race, culture, and language, and to the school contexts in which teachers develop and revise their instructional practices. A concern for issues of equity permeate the entire report but come to the fore in the discussion of school–home partnerships that take the diverse cultures of students and their families seriously and treat them as instructional resources.

Anthony and Walshaw make it clear that it is essential to view school contexts as settings for teachers’ ongoing learning. In a very real sense, these settings mediate the extent to which high quality teacher professional development will result in significant changes in teachers’ classroom practices. Anthony and Walshaw’s synthesis documents that mathematics instruction that places students’ reasoning at the center of instructional decision making is demanding, uncertain, and not reducible to predictable routines. The available evidence indicates that a strong network of colleagues constitutes a crucial means of support for teachers as they attempt to cope with these uncertainties and the loss of established routines. Consequently, there is every reason to expect that improvement in teachers’ instructional practices and student learning will be greater in schools where mathematics teachers participate in learning communities whose activities focus on central mathematical ideas and how to relate them to student reasoning. The value of teacher learning communities in turn foregrounds the critical role of the principal as an instructional leader.

Historically, teaching and school leadership have been loosely coupled, with the classroom being treated as the preserve of the teacher while school leaders managed around instruction. Recent research findings demonstrate the limitations of this type of school organisation
in supporting the improvement of teaching on any scale. These findings also indicate that principals can play a key role in supporting the emergence of a shared vision of what effective mathematical pedagogy looks like and in supporting teacher collaboration that focuses on challenges central to the development of effective pedagogy. This alternative type of school organization is characterized by reciprocal accountability. Teachers are accountable to principals for developing increasingly effective pedagogical practices and principals are accountable to teachers to create opportunities for their ongoing learning. Changes of this type in the relations between teachers and school administrators are far reaching and might be viewed as too radical. It is, however, sobering to note that previous large-scale efforts to improve the quality of classroom instruction have rarely produced lasting changes in teachers’ practices. Research into educational leadership and policy indicates that this history is due in large part to the failure to take into account the institutional settings in which teachers develop and refine their instructional practices.

The broader policy and leadership literature strongly indicates that the improvement of mathematics instruction on the scale being attempted in New Zealand is not simply a matter of providing high quality teacher professional development. It also has to be framed as a problem for schools as educational organizations that structure the institutional settings in which teachers develop and revise their instructional practices. My reading of this Best Evidence Synthesis of Effective Pedagogy in Mathematics/Pāngarau is that Anthony and Walshaw have distilled valuable lessons from the available research, thereby positioning New Zealand educators to succeed where others have failed.

Paul Cobb
Professor of Mathematics Education
Vanderbilt University, Tennessee

Note: The second Hans Freudenthal Medal of the International Commission on Mathematical Instruction (ICMI) was awarded to Professor Paul Cobb in 2005, “whose work is a rare combination of theoretical developments, empirical research and practical applications. His work has had a major influence on the mathematics education community and beyond.”

**Early Childhood Education**

This Best Evidence Synthesis of Effective Pedagogy in Mathematics/Pāngarau is a ‘must read’ for those in the early childhood sector who want an insight into what effective mathematical pedagogy looks like in an early childhood service. The synthesis acknowledges the vital role that quality early childhood education plays in the mathematical development of infants and young children. It also provokes early childhood teachers to reflect on practice: their mathematical awareness of the environment, the depth of their mathematical knowledge, and the importance of effective teaching and learning strategies that will support children’s optimal engagement in mathematical experiences. The extensive, wide-ranging research information is effectively balanced by vignettes which involve the reader in meaningful mathematical experiences that illustrate the possibilities for supporting mathematical learning. Effective distribution of the synthesis would enhance teaching and learning outcomes in early childhood services.

Faith Martin
Director, Massey Child Care Centre
NZEI Te Riu Roa

NZEI Te Riu Roa welcomes this Best Evidence Synthesis of Effective Pedagogy in Mathematics/Pāngarau, particularly as it takes for its starting point the assertion that “all children can learn mathematics”. This key message is at the heart of every teacher’s commitment to the mathematical learning of his or her students.

The synthesis recognises the complexity of teaching, particularly given the diverse learning needs of the students in our classrooms and centres and the necessity for specialised knowledge of mathematics. But the writers consistently underline the power that teachers have to make a difference: “It is what teachers do, think and believe (that) significantly influences student outcomes.”

A teacher’s role, whether in a school or a centre, includes the design of activities that help students to construct meaning and think for themselves. To achieve such outcomes, teachers need to appreciate the part that mathematics plays in the world around them, what the big mathematical ideas are, and how the concepts that they teach fit in with those ideas. They need to know how to teach knowledge and skills, how to match new learning with students’ prior knowledge, and which activities effectively encourage understanding and learning. Teachers also need to be conscious of developing attitudes and values. They need to create opportunities for their students to develop a critical eye and, in the context of this synthesis, a critical mathematical eye.

The primary purpose of the synthesis is to identify evidence that links pedagogical practice with effective mathematics outcomes for students. To achieve this, the writers have drawn on national and international research that contributes to our understanding of what works in mathematics education.

When reviewing the synthesis in its draft form, NZEI teachers were particularly pleased to read the chapter, Mathematics Practices Outside the Classroom, which they saw as contributing to a constructive environment and encouraging of good practice. The synthesis explores ways in which parents can contribute to their children’s mathematical development and ways in which schools can strengthen links with the home. If teachers are to successfully fulfil expectations, such links are likely to be vital. Teachers were also pleased to see the importance of school leadership recognised.

NZEI sees the Effective Pedagogy in Mathematics/Pāngarau BES as being of great benefit to teachers, teacher educators, and policymakers. The research identified in the synthesis, together with the case studies and vignettes, has the potential to stimulate much constructive professional discussion. To maximise its potential for teachers, it will need to be accompanied by professional learning opportunities and time for reflection and discussion in the school or centre setting.

Irene Cooper
National President
Te Manukura
NZEI Te Riu Roa
Post Primary Teachers’ Association

Tēnā koutou, tēnā koutou, tēnā tatou katoa.

PPTA welcomes this Best Evidence Synthesis of Effective Pedagogy in Mathematics/Pāngarau. It is the result of a very thorough process, inclusive of the expertise of practitioners. The final report reflects and caters to their realities, and provides some very interesting and thought-provoking reading for teachers themselves, and for those involved in the pre-service and in-service education of mathematics teachers. At the same time, the research highlights the shortage of outcomes-linked research evidence specific to secondary school mathematics teaching and we hope that as a result of this BES, New Zealand researchers will step up to fill this gap.

Debbie Te Whaiti
President
New Zealand Post Primary Teachers’ Association

Teacher Educators

The Best Evidence Synthesis of Effective Pedagogy in Mathematics/Pāngarau succeeds in providing a systematic treatment of relevant outcomes-based evidence for what works for diverse learners in the New Zealand education system. One of the strengths of the document is the central positioning given by its authors to a broad notion of diversity.

Teacher educators, both initial and ongoing, will find that the BES is an invitation to engage—as teachers and as researchers—with a wide range of national and international studies. The document succeeds in preserving the complexity of pedagogical approaches through careful structuring and presentation. Well chosen classroom vignettes capture the essence of pedagogical issues for use in initial and ongoing teacher education. The CASEs are likely to prove particularly valuable for teachers by demonstrating how research can inform classroom practice.

The BES also presents a challenge to New Zealand researchers by identifying areas in which there is a paucity of outcomes-based evidence. Such evidence is scarce for Māori-medium mathematics classrooms. The senior secondary area is generally not well represented and a wider range of early childhood contexts needs to be investigated. The CASEs highlight for teacher educators the possibilities of writing up research projects undertaken as part of ongoing teacher education initiatives, and encourage them to gather further evidence to support practice.

The importance to mathematics education of the outcomes-based research evidence represented in this synthesis cannot be overstated. It is to be hoped that the value of the Iterative BES programme is widely recognised, and that it has the impact on policy and practice that it ought.

Joanna Higgins
Director, Mathematics Education Unit and Associate Director, Jessie Hetherington Centre for Educational Research
Victoria University of Wellington
Māori-medium Mathematics

E nga mana, e nga reo, tēnā koutou katoa.

For the last 20 years, the teaching of pāngarau (mathematics) has played a significant role in the revitalisation of te reo Māori. The Effective Pedagogy in Mathematics/Pāngarau BES recognises the close relationship that exists between language and the learning and teaching of mathematics.

The BES identifies a range of major considerations and challenges for teachers and all those involved in Māori-medium education. The research makes it clear that mathematical outcomes for students are affected by a complex network of interrelated factors and environments, not just individual preferences or the language of instruction. By identifying the key elements in this network and discussing the relevant research, the writers have created what should prove a very useful resource.

The BES highlights the paucity of research into Māori-medium mathematics education, particularly in the area of teacher practice.

Tony Trinick
Māori-medium mathematics educator
Faculty of Education
The University of Auckland

Pasifika

E rima te’arapaki, te aro’a, te ko’uko’u te utuutu, ‘iaku nei.
Under the protection of caring hands there’s feeling of love and affection.

The Best Evidence Synthesis of Effective Pedagogy in Mathematics/Pāngarau has drawn together a comprehensive synthesis of evidence that relates to quality mathematics pedagogical practices. Its particular strength is that it provides stimulating and thought-provoking reading for a range of stakeholders and at the same time affirms that there is no one, specific, ‘quality’ pedagogical approach. Rather, it directs attention to many effective approaches which make a difference for all mathematics learners. The vignettes are an added strength; they make the theoretical structures they illustrate accessible to a wider audience.

The synthesis highlights the shortage of outcomes-linked research evidence concerning quality teaching and learning for Pasifika students at all levels of schooling. It also highlights the importance of a culture of care. How this translates into quality outcomes for Pasifika students requires the attention of New Zealand researchers.

Roberta Hunter
Senior Lecturer
School of Education Studies
Massey University, Albany Campus
The Effective Pedagogy in Mathematics/Pāngarau BES sets out to uncover and explain the links between what we do in mathematics education and what the outcomes are for learners. The result is a valuable resource that can be used to enhance a wide range of outcomes for diverse learners. These include the ability to think creatively, critically, strategically and logically; mathematical knowledge; enjoyment of intellectual challenge; self-regulatory, collaborative and problem-solving skills; and the disposition to use, enjoy and build upon that knowledge throughout life.

The BES reflects the outstanding scholarly work and professional leadership of co-authors Drs Glenda Anthony and Margaret Walshaw of Massey University. They are the first to use the new Guidelines for Generating a Best Evidence Synthesis and follow the collaborative development process that is central to the Iterative BES Programme. They have consulted tirelessly and responsively with a wide range of early childhood teachers, primary and secondary teachers, principals, advisers, researchers, policy workers and teacher educator colleagues from across New Zealand, and with international colleagues. The Ministry of Education acknowledges and values all these contributions—and those of the formative quality assurers, whose affirmations and challenges have been so helpful in optimising the quality and potential usefulness of this BES.

The BES celebrates and returns to early childhood educators, teachers, teacher educators and researchers a record of their professional work, highlighting the complexity of that work, and suggesting how research evidence can be a valuable resource to inform their ongoing work and that of their colleagues. From the first vignette explaining how mathematical learning can be embedded in waiata (Māori song) and dance, the vignettes bring children’s learning in mathematics to life. The underlying explanations and theoretical findings have the power to inform practice in ways that are relevant and responsive to the learners in any particular centre or classroom.

The challenge now is for us all is to use this resource in ways that will support further systemic development in mathematics education, with strengthened outcomes for diverse learners. In many cases, the BES will affirm what is already happening, but it will be the points of challenge that take us forward. Individual teachers have already engaged with the BES in its draft form, and some report remarkable insights and developments in their practice. But it is only through the wider and systemic development of the conditions that support effective practice for diverse learners that improvements will proliferate and become self-sustaining. The findings emerging from the outcomes-linked professional learning and development BESs1,2 should be an invaluable resource in determining how to generate changed practice on such a scale.

Many teachers and early childhood educators have indicated that they want to read this BES for themselves, and to do this they need time. They need time to read, discuss and consider how they can use relevant BES findings in response to diagnostic information about the mathematical understandings of the children and young people they teach. They also need time to participate in professional learning communities. The Teacher Professional Learning and Development BES3 finds that such participation doesn’t guarantee better outcomes for students, but it is a consistent feature of teacher professional learning that does have a strong positive impact.

The same BES highlights the important role that external expertise with strong pedagogical content knowledge can play in facilitating and supporting changes in practice that impacts positively on student outcomes. Such expertise can be vital in engaging teachers’ theories and challenging problematic discourses. The findings do, however, caution that ‘experts’ need more than good intentions—in the worst-case scenario, teacher professional development can actually impact negatively on student achievement. This finding calls for careful and iterative evaluation of the effectiveness of all professional development.
The teacher education community in New Zealand has already made a foundational contribution to this BES with its engagement in the research and development reported in this BES, and its advice to the BES writers. As the Teacher Professional Learning and Development BES will show, some of our most effective professional development has been taking place as part of the Numeracy Development Projects (NDP)—with effect sizes twice those attained in England. The primary and early childhood teachers’ union, NZEI, confirms what the evaluation reports have been saying: that teachers who have been involved in the NDP value the transformational experiences this professional learning has afforded them. Two teachers from a Hawkes Bay school explained to me recently that, as a result of professional learning undertaken through the NDP, they have changed the way they work across the curriculum—they now listen more, are more diagnostic, and they place much more emphasis on children articulating and sharing their learning strategies. The dynamic, reflective, nation-wide learning community of researchers, teacher educators, teachers, and other educators created by the NDP and its Māori-medium counterpart, Te Poutama Tau, has been inspirational for BES. If the mathematics BES is to serve New Zealand education well, the teacher education and research communities must make it a ‘living’ BES by building on the powerful insights and exemplars it makes available, addressing the gaps, and ensuring a cumulative and increasingly dynamic shared knowledge base about what works for learners in New Zealand education. To assist in this collaborative work, the New Zealand Council for Educational Research is creating a database of relevant New Zealand education theses. It has already built a database to support this document, with live links to the electronic version so that readers can quickly access either the full text or bibliographic details for some of the most helpful articles that have informed the synthesis. These links are also listed in the print version.

It is our hope that this BES will stimulate readers to let the Iterative Best Evidence Synthesis Programme know of other/new research and development that should feature in future iterations of the synthesis. Such research needs to clearly document demonstrated or triangulated links to student outcomes (see the Guidelines for Generating a Best Evidence Synthesis Iteration, found on the BES website), and preferably show larger positive impacts on desired outcomes for diverse learners. We are especially seeking studies of research and development in New Zealand contexts, but we are also interested in information on overseas studies that show particularly large impacts on diverse learners. Please send details to best.evidence@minedu.govt.nz.

In the New Zealand context, where schools and centres are self managing, principals and centre leaders have a critical role to play in supporting their staff to realise the potential of this BES. The Teacher Professional Learning and Development BES indicates that, in the case of the most effective school-based interventions, principals and others in leadership roles have actively supported the development of a learning culture amongst their teachers.

For centuries, societies have required their education systems to sort children into successes and failures. Knowledge societies, such as our own, require much more. Our challenge is to ensure that all our children flourish as learners, strong in their own identities, and confident global citizens.

To achieve such goals, we need to value, build upon, and go beyond the craft practice traditions that require each teacher to ‘rediscover the wheel’. The Effective Pedagogy in Mathematics/Pāngarau BES has been designed to serve as a resource and catalyst for strengthened practice, innovation, and systemic learning. By using it, and by making learner outcomes our touchstone, we can work together to give our children a mathematics education that prepares them well for the opportunities and challenges that will be their future.

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3 Ibid.

4 Timperley et al., to be published 2007.


6 Timperley et al., to be published 2007.


Authors’ Preface

What is a Best Evidence Synthesis in Mathematics?

A best evidence synthesis draws together available evidence about what pedagogical approaches work to improve student outcomes in Mathematics/Pāngarau. This synthesis is part of the Iterative Best Evidence Synthesis (BES) Programme established late in 2003 by the Ministry of Education to deepen understanding of what works in education. The programme involves policy, research, and practice in collaborative knowledge building, aimed at maximising desirable outcomes for the diverse learners in the New Zealand education system.

This best evidence synthesis in Mathematics/Pāngarau plays a key role in knowledge building for New Zealand education. As a capability tool, it identifies, evaluates, analyses, and synthesises what the New Zealand evidence and international research tell us about quality mathematics teaching. It shows us how different contexts, systems, policies, resources, approaches, practices, and influences all impact on learners in different ways. Importantly, it illuminates what the evidence suggests can optimise outcomes for diverse mathematics learners.

The importance of dialogue

The development of this BES has been shaped by the Guidelines for Generating a Best Evidence Synthesis Iteration (Alton-Lee, 2004) and informed by dialogue amongst policy makers, educators, researchers and practitioners. Right from the very early stages of its development, the health-of-the-system perspective taken in this synthesis has ensured that we have listened to and responded to the viewpoints of a wide range of constituencies. Our interactions with these multiple communities have revealed to us the key roles that infrastructure, context, settings, and accountabilities play in a system that is functioning effectively for all its learners. Our various stakeholders have challenged us not only to produce better and more relevant educational research but to consider how this knowledge base might best be used. It is our hope that this discussion across sectors will be ongoing.

We have received a strong and positive response to the best evidence synthesis work from New Zealand’s primary and post-primary teacher associations. Both have reported on how helpful the synthesis is to their core professional work. For example, the New Zealand Educational Institute (NZEI) writes: “In our view, the writers have drawn on national and international research which contributes to an understanding of what works in mathematics education; they have identified the significance of the context and ways in which to strengthen practice … We liked the … underpinning view that all children can learn mathematics” (p. 2). The representative for the Post Primary Teachers’ Association at the Quality Assurance Day is reported as saying: “There are numerous wonderful ideas in the synthesis, and I found myself repeatedly jolted into possibilities for my own classroom resources.” In addition, a group of initial and ongoing mathematics teacher educators have welcomed the “sophisticated treatment of diversity” and the way in which “the complexity of pedagogical approaches is preserved” (Victoria University of Wellington College of Education, 2006, p. 1).

Writing for multiple audiences

Our task was to make the findings of the synthesis accessible to and of benefit to a range of educational stakeholders. At one level of application, it is intended to provide a strengthened basis of knowledge about mathematics pedagogical practices in New Zealand today. The evidence it produces is expected to inform teacher educators within the discipline of mathematics education about effective pedagogical practice. At another level, the synthesis attempts to make transparent to policy makers and social planners an evidential basis for quality pedagogical approaches in mathematics. At a third level, the synthesis is expected to benefit practitioners and assist them in doing the best possible job for diverse learners in their classrooms.
Our approach to the “almost overwhelming task” (Cobb, 2006) of writing with several levels of application in mind has been to draw on both formal and informal approaches. We have offset the ‘academic’ language of the BES by including a series of vignettes that expand upon broad findings. We have received feedback from a range of sources that these vignettes bring the reality of classroom life to the fore and, in particular, do not minimise the complexities of actual practice. We hope that researchers, policy makers and practitioners alike will see in the vignettes theoretical tools that have been adapted and used by actual teachers.

The BES as a catalyst for change

This best evidence synthesis in mathematics does more than synthesise and explain evidence about what works for diverse learners. By bringing together rigorous and useful bodies of evidence about what works in mathematics, the project plays an important function as a catalyst for change. It is designed to help strengthen education policy and educational development in ways that effectively address both the needs of diverse learners and patterns of systemic underachievement in New Zealand education. It is written with the intent of stimulating activity across practitioners, policy makers, and researchers and so to strengthen system responsiveness to educational outcomes for all students.

The writers anticipate that reflection on the findings will lead to sustainable educational development that has a positive impact on learners. It will create new insights into what makes a difference for our children and young people. Reflection on the findings will also spark new questions and renewed, fruitful engagement with mathematics education. These new questions, in turn, will render the BES a snapshot in time—provisional and subject to future change.

Key features

Key features of the BES are:

- Its teacher orientation. Its view is towards a strengthened basis of knowledge about instructional practices that make a difference for diverse groups of learners.
- Its cross-sectoral approach. Its scope takes in the teaching of children in early childhood centres through to the teaching of learners in senior secondary school classrooms.
- Its inclusiveness. It documents research that reveals significant educational benefits for a wide range of diverse learners. It pays particular attention to the mathematical development of Māori and Pasifika students and documents research that captures the multiple identities held by New Zealand learners.
- Its breadth of search coverage. It reports on the characteristics of effective pedagogy, following searches through multiple national databases and inventories as well as masters’ projects and theses. It provides comprehensive information about effective teaching as evidenced from small cases, large-scale explorations, and short-term and longitudinal investigations.
- Its local character. It makes explicit links between claims and bodies of evidence that have successfully translated the intentions and spirit of the Treaty of Waitangi. It identifies research relevant to the particular conditions and contexts in New Zealand, both in mathematics education in particular and in education in general, in relation to the principles and goals of Te Whāriki for early childhood settings and of The Curriculum Framework, for teachers in English or Māori-medium settings.
- Its global linkages. It connects local sources with the international literature. It identifies important Australian and international work in the area and evaluates that wide-ranging resource in relation to similarities and differences in cultures.
populations and demographics between the country of origin and New Zealand.

- **Its responsiveness to concerns about democratic participation.** It heeds the concern about the development of competencies that equip students for lifelong learning. This orientation coincides with the national mathematics curriculum objective of developing those knowledges, skills, and identities that will enable students to meet and respond creatively to real-life challenges.

- **Its quality assurance measures.** It is guided by principles of transparency, accessibility, relevance, trustworthiness, rigour, and comprehensiveness. These principles form the backdrop to the selection and systematic integration of evidence.

- **Its strategic focus on policy and social planning.** It uses a health-of-the-system approach to address one of the most pressing problems in education, provide a direction for future growth, and push effective teaching beyond current understandings.

- **Its provisional nature.** The project is an important knowledge-building tool, creating new insights from what has gone before, and will be updated in the light of findings from new studies. The findings are, above all, ‘of the moment’ and open to future change.

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**References**


Executive Summary

The Effective Pedagogy in Mathematics/Pāngarau: Best Evidence Synthesis Iteration (BES) was funded by a Ministry of Education contract awarded to Associate Professor Glenda Anthony and Dr Margaret Walshaw at Massey University. The synthesis is part of the Iterative Best Evidence Synthesis (BES) Programme, established by the Ministry of Education in New Zealand, to deepen understanding from the research literature of what is effective in education for diverse learners. The synthesis represents a systematic and credible evidence base about quality teaching in mathematics and explains the sort of pedagogical approaches that lead to improved engagement and desirable outcomes for learners from diverse social groups. It marks out the complexity of teaching and provides insight into the ways in which learners’ mathematical identities and accomplishments are occasioned by effective pedagogical practices.

The search of the literature focused attention on different contexts, different communities, and multiple ways of thinking and working. Priority was given to New Zealand research into mathematics in early childhood centres and schools, both English- and Māori-medium. Personal networks enhanced the library search and facilitated access to academic journals, theses and reports, as well as other local scholarly work. The New Zealand literature was complemented by reputable work undertaken in other countries with similar population and demographic characteristics. Indices, both print and electronic, were sourced, and the search covered relevant publications within the general education literature as well as specialist educational areas. In the end, 660 pieces of research, ranging from very small, single-site studies to large scale, longitudinal, experimental studies, found their way into the report.

Key findings highlight practices that relate specifically to effective mathematics teaching and to positive learning and social outcomes in centres/kōhanga and schools/kura. The findings stress the importance of interrelationships and the development of community in the classroom. They also reveal that both the cognitive and material decisions made by teachers concerning the mathematics tasks and activities they use, significantly influence learning. The findings demonstrate the importance of children’s early mathematical experiences and stress that constituting and developing children’s mathematical identities is a joint enterprise of teacher, centre/school, and family/whānau.

Key findings

In this section, key findings are organised and presented according to five themes: the key principles underpinning effective mathematics teaching, the early years, the classroom community, the pedagogical task and activity, and educational leadership and centre–home and school–home links.

Key principles underpinning effective mathematics teaching

Teachers who enhance positive social and academic outcomes for their diverse students are committed to teaching that takes students’ mathematical thinking seriously. Their commitment to students’ thinking is underpinned by the following principles:

- an acknowledgement that all students, irrespective of age, have the capacity to become powerful mathematical learners;
- a commitment to maximise access to mathematics;
- empowerment of all to develop mathematical identities and knowledge;
- holistic development for productive citizenship through mathematics;
- relationships and the connectedness of both people and ideas;
- interpersonal respect and sensitivity;
- fairness and consistency.
The early years

Young children are powerful mathematics learners. Quality teaching guarantees the development of appropriate relationships and support as well as an awareness of children’s mathematical understanding. Research has consistently demonstrated how a wide range of children’s everyday activities, play and interests can be used to engage, challenge and extend children’s mathematical knowledge and skills. Researchers have found that effective teachers provide opportunities for children to explore mathematics through a range of imaginative and real-world learning contexts. Contexts that are rich in perceptual and social experiences support the development of problem-solving and creative-thinking skills.

There is now strong evidence that the most effective settings for young learners provide a balance between opportunities for children to benefit from teacher-initiated group work and freely chosen, yet potentially instructive, play activities. Opportunities for learning mathematics typically arise out of children’s everyday activities: counting, playing with mathematical shapes, telling time, estimating distance, sharing, cooking, and playing games. Teachers in early childhood settings need a sound understanding of mathematics to effectively capture the learning opportunities within the child’s environment and make available a range of appropriate resources and purposeful and challenging activities. Using this knowledge, effective teachers provide scaffolding that extends the child’s mathematical thinking while simultaneously valuing the child’s contribution.

The classroom community

Research has shown that opportunities to learn depend significantly on the community that is developed within centres and classrooms. Thus, people, relationships, and classroom environments are critically important. Whilst all teachers care about student engagement, research quite clearly demonstrates that pedagogy that is focused solely on the development of a trusting climate does not get to the heart of what mathematics teaching truly entails. Teachers who truly care about their students have high yet realistic expectations about enhancing students’ capacity to think, reason, communicate and reflect upon their own and others’ understandings. Research has provided conclusive evidence that effective teachers work at developing inclusive partnerships, ensure that the ideas put forward by learners are received with respect and, in time, become commensurate with mathematical convention and curricular goals.

Studies have provided conclusive evidence that teaching that is effective is able to bridge students’ intuitive understandings and the mathematical understandings sanctioned by the world at large. Language plays a central role. Mathematical language involves more than vocabulary and technical usage; it encompasses the ways that expert and novice mathematicians use language to explain and to justify concepts. The teacher who has the interests of learners at heart ensures that the home language of students in multilingual classroom environments connects with the underlying meaning of mathematical concepts and technical terms. Teachers who make a difference are focused on shaping the development of novice mathematicians who speak the precise and generalisable language of mathematics.

Mathematics teaching for diverse learners creates a space for the individual and the collective. Whilst many researchers have shown that small-group work can provide the context for social and cognitive engagement, others have cautioned that students need opportunities and time to think and work quietly away from the demands of a group. There is evidence that some students, more than others, appear to thrive in class discussion groups. Many students, including limited-English-speaking students, are reluctant to share their thinking in class discussions. Research has also shown that an over-reliance on grouping according to attainment is not necessarily productive for all students. Teachers who teach lower streamed classes tend to follow a protracted curriculum and offer less varied teaching strategies. This pedagogical...
practice may have a detrimental effect on the development of a mathematical disposition and on students’ sense of their own mathematical identity.

**Pedagogical tasks and activities**

From the research, it is evident that the opportunity to learn is influenced by what is made available to learners. For all students, the ‘what’ that they do is integral to their learning. The ‘what’ is the result of sustained integration of planned and spontaneous learning opportunities made available by the teacher. The activities that teachers plan, and the sorts of mathematical inquiries that take place around those activities, are crucially important to learning. Effective teachers plan their activities with many factors in mind, including the individual student’s knowledge and experiences, and the participation norms established within the classroom. Extensive research in this area has found that effective teachers develop their planning to allow students to develop habits of mind whereby they can engage with mathematics productively and make use of appropriate tools to support their understanding.

Choice of task, tools, and activity significantly influences the development of mathematical thinking. Quality teaching at all levels ensures that mathematical tasks are not simply ‘time fillers’ and is focused instead on the solution of genuine mathematical problems. The most productive tasks and activities are those that allow students to access important mathematical concepts and relationships, to investigate mathematical structure, and to use techniques and notations appropriately. Research provides sound evidence that when teachers employ tasks for these purposes over sustained periods of time, they provide students with opportunities for success, they present an appropriate level of challenge, they increase students’ sense of control, and they enhance students’ mathematical dispositions.

Effective teaching for diverse students demands teacher knowledge. Studies exploring the impact of content and pedagogical knowledge have shown that what teachers do in classrooms is very much dependent on what they know and believe about mathematics and on what they understand about the teaching and learning of mathematics. Successful teaching of mathematics requires a teacher to have both the intention and the effect to assist pupils to make sense of mathematical topics. A teacher with the intention of developing student understanding will not necessarily produce the desired effect. Unless teachers make good sense of the mathematical ideas, they will not have the confidence to press for student understanding nor will they have the flexibility they need for spotting the entry points that will move students towards more sophisticated and mathematically grounded understandings. There is now a wealth of evidence available that shows how teachers’ knowledge can be developed with the support and encouragement of a professional community of learners.

**Educational leadership and links between centre and home/school**

Facilitating harmonious interactions between school, family, and community contributes to the enhancing of students’ aspirations, attitudes, and achievement. Research that explores practices beyond the classroom provides insight into the part that school-wide, institutional and home processes play in developing mathematical identities and capabilities. There is conclusive evidence that quality teaching is a joint enterprise involving mutual relationships and system-level processes that are shared by school personnel. Research has provided clear evidence that effective pedagogy is founded on the material, systems, human and emotional support, and resourcing provided by school leaders as well as the collaborative efforts of teachers to make a difference for all learners.

Teachers who build whānau relationships and home–community and school–centre partnerships go out of their way to facilitate harmonious interactions between the sectors. There is convincing evidence to suggest that these relationships influence students’ mathematical development. The home and community environments offer a rich source of mathematical experiences on which to build centre/school learning. Teachers who collaborate with parents, families/whānau and
community members come to understand their students better. Parents benefit too: through their purposeful involvement in school/centre activities, by assisting with homework, and in providing suitable games, music and books, they develop a greater understanding of the centre’s or school’s programme. Their involvement also provides an opportunity to scaffold the learning that takes place within the centre or school.

**Overall key findings**

This Best Evidence Synthesis examines the links between pedagogical practice and student outcomes. Consistent with recent theories of teaching and learning, it finds that quality teaching is not simply a matter of ‘knowing your subject’ or ‘being born a teacher’.

Sound subject matter knowledge and pedagogical content knowledge are prerequisites for accessing students’ conceptual understandings and for deciding where those understandings might be heading. They are also critical for accessing and adapting task, activities and resources to bring the mathematics to the fore.

The importance of building home–community and school–centre partnerships has been highlighted in a number of studies of effective teaching.

Early childhood centre researchers have provided evidence that the most effective settings offer a balance between opportunities for children to benefit from teacher-initiated group work and freely chosen, yet potentially instructive, play activities.

Within centres and classrooms, effective teachers care about their students and work at developing interrelationships that create spaces for learners to develop their mathematical and cultural identities.

Extensive research on task and activity has found that effective teachers make decisions on lesson content that provide learners with opportunities to develop their mathematical identities and their mathematical understandings.

Studies have provided conclusive evidence that teaching that is effective is able to bridge learners’ intuitive understandings and the mathematical understandings sanctioned by the world at large.

**Gaps in the literature and directions for future research**

The synthesis provides research information about effective mathematics teaching. Although the scope of researchers’ studies is broad and far-reaching, a number of gaps in the literature are apparent. Research has so far provided only limited information about effective teaching in New Zealand at the secondary school level. Additionally, there is little reported research that focuses on quality teaching for Pasifika students. Few researchers in New Zealand are exploring mathematics in early childhood centres. The New Zealand literature lacks longitudinal, large-scale studies of teaching and learning. Also missing are studies undertaken in collaboration with overseas researchers. Such research is crucially important for understanding teachers’ work and the impact of curricular change. The scholarly exchange of ideas made possible through joint projects with the international research community contributes in numerous ways to the capability of our local researchers.

It is important to keep in mind that, as a knowledge building tool, the synthesis provides insights based on what has gone before. A snapshot in time, it is subject to change as new kinds of evidence about quality teaching become available. Important mathematics initiatives are underway in New Zealand schools and centres. The Numeracy Development Projects, new assessment methods, projects involving information technology, and a greater focus on statistics in the curriculum are just three examples of changes that are currently taking place. All new initiatives require research that monitors and evaluates their introduction and ‘take up’ by centres/schools and the changes in teaching and learning that take place as a result. Such research is necessary to guide future directions in schools, educational policy, and curriculum design.
7. A Fraction of the Answer

Introduction

The evidence-based discussions in the preceding chapters serve to highlight the complexity of the learning and teaching process. In this chapter, we present extracts, which we call CASEs, from research studies situated in the teaching and learning of fractions. By grouping the CASEs around the domain of fractions (we include proportion and ratio within this domain), we hope to weave the threads of the preceding chapters into a whole that will be particularly helpful for initial and ongoing teacher education.

In presenting these CASEs, our aim is to capture the principal activities of teaching: (a) creating and supporting a learning community; (b) analysing and building on students' existing conceptions; (c) making sense of the mathematical ideas to be taught; and (d) selecting/using tasks that foster students' conceptual advances. At the same time, we also attempt to capture some of the principal activities of learning: (a) participating in a range of mathematical practices; (b) reflecting on one's learning processes; and (c) collaborating with others in sense making. This approach stems from our view that successful teaching involves processes that teachers and students have to work through together. "Teaching is not something that is merely 'done' to learners. [Students] are active participants in classrooms and the transactional nature of teaching and learning means that [students] as much as teachers shape the development of lessons" (Askew, in press).

Why have we chosen fractions? As many teachers know from first-hand experience, fractions are one of the most complex mathematical domains that students encounter during their school years (Davis, Hunting, & Pearn, 1993). The difficulties that New Zealand students experience with fractions have been highlighted in reports evaluating the Numeracy Development Project and Te Poutama Tau (e.g., Christensen, 2004; Young-Loveridge, 2005).

Fractions constitute a body of knowledge considered essential not only for higher mathematics but also for everyday life. It is our understanding of fractions that makes it possible for us to contemplate ‘how much?’ as opposed to ‘how many?’ Understanding of fractions signals a shift from reasoning additively to reasoning multiplicatively. This development begins when young children experience sharing situations (Hunting & Sharpley, 1988) and—in the school context—culminates with ratio and proportional thinking (Thompson & Saldanha, 2003).

While early research into students’ understanding of fractions tended to document their poor performance and their misconceptions (e.g., Hart, 1988), more recent research has explored the possibilities that exist within instructional settings for the creation and development of fractional knowledge. We have selected CASEs from research to exemplify aspects of effective pedagogy: pedagogy that makes a difference to student outcomes. In some studies, the differences are measured by pre- and post-tests, sometimes by contrast with the performance of students in other classes (e.g., CASE 4). In other CASEs, change is indicated by the success of one or two learners who are beginning to participate in mathematical practices, resolve cognitive conflict, grow in understanding, or develop metacognitive awareness (e.g., CASE 7).

In each of the selected CASEs, opportunity and space for learning are key factors. A supportive environment and established social and sociomathematical norms encourage students to reflect on their learning with an expectation that they will be able to make sense of the mathematics involved. Teacher awareness and responsiveness, developed through critical listening to student responses and thinking, are also key features of the selected CASEs.

In selecting exemplary CASEs, it is not our intention to minimise or hide the everyday complexity of the teaching and learning environment. Several of the CASEs are from studies that involve teacher professional development. The full reports of these studies often include teacher commentary on the real-time challenges associated with task selection, managing group work,
and facilitating meaningful discussion—including dealing with the unintelligible in classroom discussions. O’Connor (2001) acknowledges that teachers, when presented with accounts of classroom discussions conducted by talented and skilful teachers, are often concerned with what is not in the transcript: “What were the other students doing while the teacher attended to a particular group of students?”; “Why didn’t she just tell them about $x$ or $y$?”; or “Why did she ignore that student’s incorrect answer?” So, in reviewing each of the following CASEs, the reader might reasonably ask, “What is left out of the account?”

In each CASE, the presented episodes are adapted from research accounts and are therefore necessarily selective and somewhat brief. Although chosen to exemplify quality pedagogical practices, the CASEs are not intended, by themselves, to provide a list of ‘good ideas’ for teaching fractions. Rather, the expectation is that they will stimulate critical thought about effective pedagogical practices for mathematics in general and for fractions in particular.

**Building on learners’ prior knowledge and experiences**

Prior to formal schooling, many opportunities for learning mathematics can be found in children’s everyday experiences—in the home, the community, or centre. CASE 1 exemplifies the supporting role of the adult. Building on a relationship of trust, the adult is able to engage with the child in sustained, shared thinking involving an imaginative and experientially real context.

### CASE 1: Cookies

*(from Sharp, Garofalo, and Adams, 2002)*

Mathematics teaching for diverse learners:

- demands an ethic of care;
- creates a space for the individual and the collective;
- provides opportunities for children to explore mathematics through a range of imaginative and real-world contexts;
- provides for both planned and spontaneous/informal learning.

Through experiences such as ‘sharing’, young children develop intuitive fractional knowledge in which they combine thought, informal language, and images (Kieren, 1988).

**Targeted outcomes**

Informal knowledge of fractions developed through context-based sharing situations.

**Learning context**

Leah, who is almost five, and Joe, an adult, often play games while driving to Leah’s preschool in their truck. Leah is engaged in a sustained shared-thinking episode as part of a game that she regularly plays with Joe, which she calls ‘Kids and Cookies’.

**Task and activity**

- **Joe:** Hey Leah, what do you want to play today?
- **Leah:** Let’s play Kids and Cookies.
- **Joe:** OK. What if you had 4 cookies and 3 kids? How would you share them?
- **Leah:** One, one, one, and then there is one left. Then they each get one third, one third, one third.
- **Joe:** So, how much does each kid get?
- **Leah:** They get one whole one and one third.
- **Joe:** What if you had 5 cookies and 3 kids? How could you share the cookies?
- **Leah:** One, one, one. Then there’s two more left. OK. Then they get a third, a third, a third, and then a third, a third, a third.
Joe: So how much do they each get?
Leah: They get one whole and two thirds.
Joe: What if you had 7 cookies and 4 kids?
Leah: That’s a hard one, maybe I can’t do it.
Joe: Think about what you did to solve the other two.
Leah: Whole, whole, whole, whole, then there’s three more left. Um, three more cookies left. Then you break up one into halves, then there are two left. And, another into half, half. Break the last one into quarter, quarter, quarter, quarter.
Joe: Great! How much does each kid get?
Leah: One whole, one half, and one quarter.

Joe and Leah played the game for a few minutes, several times a week, with Joe varying the numbers so that Leah was able to resolve more and more complex situations. Sometimes Joe would ask Leah to find two ways to share the cookies (e.g., four cookies could be shared among six children by splitting each cookie into sixths and giving each child four-sixths, or by giving each one-half of a cookie and one-sixth).

Learner outcomes
By comparing different solution strategies for problems, Leah developed an understanding of equivalent fractions. With repeated exposure to the game, Leah’s ideas were both valued and challenged. She was able to build on her existing conceptual understanding of fractions and operational sense of whole number to develop a procedure that would foreground operations with fractions.

Quality pedagogy
Leah’s growth in understanding of fractions was assisted through:

- interaction with a supportive adult. Joe’s questions incorporated ideas about sharing and used Leah’s informal language.
- activities situated in experientially relevant contexts. Leah’s conceptual knowledge of fractions grew from encounters with whole-number division.
- instruction that built on her informal knowledge. This gave Leah access to participation, allowing her to make contributions that were personally meaningful.

Bridge to school
Children may well enter school with a rich bank of informal or intuitive understanding of rational number concepts and procedures, based on their activities in their personal environment. It is through these activities that students develop thinking tools and imagery for the construction of important knowledge about rational numbers.

**Positioning participants as proficient learners**

In chapter 4, we found that teachers who produce effective classroom communities seek to develop interrelationships that create spaces for students to develop their mathematical identities. In caring, teachers developed a culture that did not minimise individuals’ experiences and contributions within the classroom. Students were trusted with responsibility for themselves and their learning and were provided with opportunities to exercise this responsibility (Angier & Povey, 1999). In CASE 2, the focus is on two low-performing students’ experiences in a series of lessons based on equal-sharing fraction tasks. Empson (1999) takes the data from a successful classroom intervention study involving early fraction learning and reanalyses it from a participation perspective, in order to unpack how it is that these two students profited from their classroom experience, “not despite the cognitive or social skill they may have lacked but because of the way their teacher orchestrated their participation in solving and discussing problems” (p. 305).
CASE 2: Low-performing students’ participation
(from Empson, 2003)

Mathematics teaching for diverse learners:
• demands an ethic of care;
• demands teacher content knowledge and pedagogical content knowledge and reflecting-in-action;
• involves explicit instructional discourse;
• provides opportunities to explore mathematics through a range of relevant contexts;
• provides tasks that are problematic and have a mathematical focus;
• provides opportunities to resolve cognitive conflicts and problematic reasoning.

This case presents two low-performing students’ experiences in a first grade classroom. Explanations for student gains in fraction knowledge are analysed from the perspective of the dynamics of the instructional interactions and their consequences for the students.

Targeted learning outcome
Gains in Patrick and Pho’s understanding of fractions.

Learning context
The data are drawn from a larger case study (Empson, 1999) of a US grade 1 class involved in a five-week unit on fractions. The instruction was organised around eliciting and building on children’s informal knowledge of equal-sharing situations.

The students
Two students, Patrick and Pho, were assessed in a clinical interview as knowing least about fractions, when compared with their classmates, both at the beginning and at the end of the study. Patrick, from a middle-class family, had been informally identified by school personnel as having trouble focusing on academic tasks. Because he was being withdrawn to participate in an alternative programme, he was present for a total of 9 of the 15 fractions lessons. Pho’s family spoke English as a second language. He was present for all 15 lessons.

Pedagogical approach
Assisting students to resolve problematic reasoning is an issue faced by all teachers, especially so in fractions (Thomson & Salanha, 2004). Empson provides several vignettes that illustrate how the teacher, Ms. K, played a key role in orchestrating the participation of Patrick and Pho in the mathematical practices of the classroom community. The participant frameworks that emerged in her classroom supported Patrick and Pho to participate, when possible, in resolving problematic aspects of their reasoning.

This vignette involves a group situation. Students are solving the problem “There are 2 horses in the field. If they have one bushel with 9 apples in, how many apples would each horse get?” The interaction that follows begins after the first student explained his strategy to the group, including how he split the extra apple in half. When Ms. K called on Pho to report his solution, a predicament emerged with Pho’s declaration that he could not give the horses the extra apple:

1 Pho: There’s one more left, but I can’t give them this.
2 Ms. K: So what are you going to do with that one left?
3 Pho: Uh.
4 Ms. K: Anybody got an idea? What can we do with that other one?
5 James: I know, I know.
6 Ms. K: (To Pho) Do you know what you could do with this one?
7 Pho: (Shakes head no)
8–9 Ms. K: No. You’re not sure. Could anyone share with him what he could maybe do with that other one? James?
10 James: Cut it in half.
11 Ms. K: Could you cut it in half, Pho?
Pho: (inaudible)
Ms. K: And then how much would each horse get?
Pho: It still wouldn’t work.
Ms. K: Why won’t it work?
Pho: ’Cause, there still is one more left.
Ms. K: OK. How much is this horse [pointing to his cubes] going to get?
Pho: Four.
Ms. K: How much is this horse [pointing to second horse] going to get?
Pho: Four.
Ms. K: James, he says it is still not going to work. Would you please go down—
Pho: (Interrupting) Because four and four is eight.
James: [To Pho.] You have eight so far. Just cut that (extra apple) in half, and then each one would get eight (sic) and a half.
Pho: I still can’t get it.

In this episode Pho was positioned as a problem solver. Throughout the episode the teacher elicited help from the other children. The distribution of the role of problem solver enabled the other children to act as potential sources for problem-solving ideas. Pho, however, retained the authority to accept or reject the ideas offered based on his understanding of that idea, and in this case he was seen to reject a mathematically legitimate idea proposed by James. While the rest of the group solved a new problem, Ms. K worked with Patrick and Pho to help them arrive at the solution of partitioning the extra apple in half and verbalising how that quantity related to the whole apples (see the following interaction with Patrick).

Patrick: Hmmm.
Ms. K: What did the other kids say we could do with this other apple?
Patrick: Split it in half. But we can’t do it.
Ms. K: Well, let’s pretend it’s [holds up a linking cube] an apple. If it’s an apple could we cut it in half? [Patrick agrees.] ... So how many apples is each horse gonna get in this now?
Patrick: One half.
Ms. K: OK. And they’re gonna get these [indicating four linking cubes each] ... How much would they get?
Patrick: Five.
Ms. K: How did you figure out five apples?
Patrick: Because if we cut this [extra linking cube] in half they would each get five apples.
Ms. K: Show me. [Patrick counts four single apples each, and on the extra cube, two apples (one for each half).] Is this last piece they’re gonna get a whole apple or is that gonna be a half apple?
Patrick: Half apple.
Ms. K: Are these [four linking cubes] whole apples or are they half apples?
Patrick: Whole.
Ms. K: So how many whole, big apples are they gonna get?
Patrick: Four.
Ms. K: OK. And how many half apples?
Patrick: One. [Patrick writes ‘4 1’ on his paper.]

The interactions were structured so that Patrick and Pho revoiced the key ideas introduced by other children in the earlier exchange. This enabled the two boys to be responsible for evaluating potentially useful mathematical ideas and to begin to make the ideas their own. Throughout the interactions the teacher’s use of “what?” and “how?” questions directed the boys’ attention to the mathematically critical aspects of the solution—that is, the difference between whole apples and fractional apples as amounts. Physical materials were used as a support for thinking, rather than a literal representation (see Higgins, 2005).
In this and other episodes provided by Empson, we can see that Ms. K oriented Patrick and Pho’s participation in instruction towards problem-solving practices and towards taking on an authoritative role. A significant factor in gaining their participation was the acknowledgement and building on the task-based contribution that each boy was able to make. Empson noted that Patrick and Pho had ideas about how to solve almost all of the problems. While these ideas were sometimes partially formed, ambiguously stated, or notationally unsophisticated, they were, with the assistance from Ms. K’s scaffolding, able to be treated as part of the pool of ideas for solving the problem. The contributions of Patrick and Pho were accepted or rejected based on mathematical reasons supplied by the learning community.

In the following episode, Pho is positioned as a mathematical authority. This is the first time children are asked to solve an equal-sharing problem involving partition into thirds, a partition that, from a geometric perspective, is harder to make than partitions involving repeated halving (Pothier & Sawada, 1983): “Three children want to share seven candy bars so that everyone gets the same amount. How much would each child get?”

1–2 Marie: Because there’s—you can’t leave one over [i.e., if you make fourths, you will have an extra piece], so if you cut this one [extra cube] in half—

3–5 Ms. K: But Marie, look what Pho did [Marie looks at Pho’s paper]. Pho, she says you can’t split that in three. You think that’s right? [Inaudible answer from Pho.] What do you think guys?

6–7 Tim: If you split in three, then you would get a half that remains, a half of a candy bar that’s still there [Empson noted: he may mean a fourth].

8 Ms. K: If you split it in threes?

9 Marie: That’s what I’m talking about. You still get a quarter left.

10 Ms. K: Pho, they said that you can’t split it in threes.

11 Tim: You can split it in threes but you have a half left.

12–13 Ms. K: You’ll have a half? Look at how he split it, Tim. Does he have a half left over? [Lev goes over to look at Pho’s paper; Tim looks too.]

14 Tim: No.

15 Ms. K: [To Pho] Do you have a half left over?

16 Kaitlin: No.

17 Pho: No.

18 Tim: That’s because of the way you cut it.

19 Ms. K: He cut it differently didn’t he? Would that work, Tim, or not?

20 Tim: It would work.

21 Pho: I’ll draw it over.

22 Ms. K: Would you draw it bigger so people can see.

In this episode Pho was animated by the teacher making a mathematical claim in opposition to another apparently reasonable mathematical claim. In managing the distributed argumentation the teacher created opportunities for Pho to respond directly to Tim and Marie’s claim about the impossibility of splitting the bar in three. By relaying their statements to Pho, the teacher effectively scaffolded Pho in this role. In her contribution in lines 12–13, the teacher directed Tim to the part of Pho’s diagram illustrating the main claim, thus modelling a move Pho could have made himself. Empson claims that the explicit positioning of the competing ideas assisted the students to resolve the conflicting representations and elevated Pho’s solution to the status of a defensible claim of value (lines 20–21). Ultimately, thirds became an acceptable partition in the class. In a later interaction, Patrick was also positioned in the role of author of a partitioning strategy that formed a critical piece in an argument about equivalent fractions.

Learner outcomes

Comparison of the pre- and post-interview results documents Patrick’s and Pho’s gains in understanding. Throughout the series of lessons Ms. K positioned Patrick and Pho to make contributions to group discussions that enabled them to be animated in identity-enhancing ways—as problem solvers, solution reporters, and claim defenders. These students, although low attainers, were able to successfully engage in problem-solving processes, communicate their thinking, and build complex arguments about mathematical relationships—practices that are essential for learning and doing mathematics [Watson & De Geest, 2005].
Quality pedagogy

Empson proffers three main factors to explain how these two boys, who clearly struggled in mathematics, were supported to participate profitably in Ms. K’s classroom:

• Both Patrick and Pho were treated as competent in ways that were valued in the classroom community of practice. Contributions, based on their informal ideas, enabled the boys to be positioned as legitimate knowers (Lave & Wenger, 1991).

• The mathematical tasks posed to Patrick and Pho allowed them to make use of prior knowledge to generate new strategies. These semantically rich problems afforded a variety of strategies—in Pho and Patrick’s case, partial strategies—which provided a basis for productive interactions between teacher and students.

• Patrick and Pho had multiple opportunities to learn the value of their ideas and the practices entailed in developing and articulating those ideas. They were at all times regarded as mathematical thinkers and doers.

"Although Ms. K knew that Patrick and Pho did not, generally, understand as much about fractions as the other students, they were not animated as children who did not understand. They were instead animated as children who engage in the practice of mathematics, and consequently, as children who understood mathematics. This positioning positively affected their participation and learning” (p. 339).

Challenging tasks with a mathematical focus

As we have seen in chapter 5, high-involvement teachers like Ms. K in CASE 2 typically present challenge as desirable, treat errors as informational, provide feedback on progress, and help students resolve conflicting reasoning. High-involvement classrooms were the focus of Turner and colleagues’ (1998) study. A sixth grade teacher in this study, Ms. Adams, took a lesson on fractions and the researchers noted how she regulated the challenge of the task to match her students. Instead of altering the task goal, she adjusted the instructions until her students’ skills matched the challenge. For example, to avoid reducing the overall complexity or compromising the integrity of the task, Ms. Adams modelled strategies such as reducing fractions expressed with large numbers to equivalent fractions expressed with smaller numbers. In contrast, when a low-involvement teacher in the same study wanted her students to convert $\frac{161}{184}$ to a percentage, she took over the task: “Most of us don’t remember this, but if we want to turn this into a decimal, we would divide 184 into 161.”

Students in low-involvement classrooms typically report feelings of boredom and less positive affect, and that their skills exceed the challenges provided. Houssart’s (2002) UK study of a class of nine- and ten-year-old ‘low attainers’ illustrates how task challenge and related mathematical focus can influence students’ performance. During a six-week unit on fractions, the children in the research class completed 18 worksheets requiring low-level identification and colouring of fractions of shapes. Houssart’s observations of three boys revealed that each attempted, both privately and publicly, to extend ideas introduced by the teacher. For example, in a folding exercise to demonstrate halves, the boys speculated on whether thirds could be made in a similar way. The teacher, however, failed to acknowledge the students’ attempts to extend the task. As the unit progressed, the boys commented on the ease of the work and their behaviour became more disruptive and off-task. The teacher’s reaction was to express disappointment with the class and focus on structured worksheets. Henceforth, the instructional focus became task completion rather than the development or demonstration of mathematical understanding. As a consequence of the written work being too easy, the boys worked quickly without listening to the teacher, re-reading instructions, or seeking help. The fact that the work was so easy became, according to Houssart, a progressively stronger factor in the boys’ apparent failure. By the end of the unit, these boys were failing to complete much of the easy work and they were communicating their dissent publicly.

Superficially, this could be seen merely as bad behaviour and failure to cooperate. However, the crucial point is that the behaviour was a result of shifting classroom

The study raises two significant issues for pedagogical practice. First, when gathering evidence about student learning, teachers need to consider evidence from a range of tasks, both written and oral. And second, more significantly, with regard to task challenge, Houssart’s work contests the view that task simplification and repetition is appropriate for low attainers.

In contrast, tasks that present higher-level demands use procedures but in a way that build connections to the mathematical meaning (Stein et al., 1996). CASE 3 presents an example of a challenging task that arose spontaneously during a class activity. The mathematically rich activity invited student exploration and challenge at a range of levels appropriate to all students in the class.

### CASE 3: Flags
(from Kieren, Davis, and Mason, 1996)

Mathematics teaching for diverse learners:
- involves explicit instructional discourse;
- creates a space for the individual and the collective;
- demands teacher content and pedagogical content knowledge and reflecting-in-action;
- provides opportunities for cognitive engagement and a press for understanding;
- utilises tools as learning supports.

This case describes students’ exploration of fraction concepts using a student-generated activity, ‘fraction flags’. The case is framed from the perspective that students benefit from engaging in mathematically rich activity—activity that invites exploration and conjecture. In addition, the teacher selects classroom learning activities that are responsive to students’ knowledge and interests.

### Targeted learning outcomes
Learners view fractions as representing additive quantities and as showing multiplicative relationships.

### Learning context
This activity is derived from an exploration by two 12-year-old students, Tanya and Ellen, of the pieces from a ‘pizza fractions’ kit. The six-week unit that gave rise to the ‘flags’ activity was developed around various paper-manipulating activities: folding, cutting, comparing, rearranging, and assembling. The tools used to support learning included physical manipulation of units and fractional sub-units built from paper, mental actions on the images of fractions constructed by students, and the verbal and symbolic expressions of actions, observations, and justification.

### Task and student activity
The fractions unit centred on providing opportunities for students to build their own ideas of fractions. A key representation for investigating multiplicative notions involved folding units (standard pieces of paper) and sub-units into various numbers of parts. For example, by exploration, repeated folding could generate thirty-twoths. Another representation, the ‘pizza fractions’ kit, (assorted rectangular fractional pieces including wholes, halves, thirds, fourths, sixths, eights, twelfths, and twenty-fourths, as in figure 7.1) offers students the opportunity to develop an additive, quantitative sense of rational numbers.
Activities based around the pizza kit included student-generated pizza orders, such as $\frac{1}{4} + \frac{1}{3} + \frac{5}{12}$. Students’ arrangement of the pieces supported the development of images and understandings. Several strategies were used by students to present their results, including drawing pictures, writing out fraction phrases and sentences, and reproducing summary charts.

Within this context, Kurt, playing with some pieces from the kit while waiting for the teacher, created a flag (fig. 7.2). Prompted by another student’s enquiry as to how much of the paper was left uncovered, the teacher structured a new setting for further fraction problems.

The ‘fraction flags’ activity was introduced to the class in this way:

*Take a half piece, a twelfth piece, and two eighth pieces from your kit and make this flag (see fig. 7.3). Show it to your partner and make up some fraction questions about the flag. For example, is more of the base covered or uncovered? How much more? Use pieces from your kit to make up your own flag. Once you have done this, make up some fraction questions about it. Try to have your partners or other students in class answer your questions. All members of your group should be ready to discuss your flags and questions with the rest of the class. Remember, try to make interesting flags, but also make flags so that you can ask good fraction questions about them.*
Student outcomes

When left to design their own questions within contexts that were of interest, students developed situations that were personally challenging. For example, Tanya and Ellen made a fraction flag (fig. 7.4) by taking a half-sheet of paper and arranging smaller pieces on it, then worked out how much of the whole sheet of paper was covered.

![Fig. 7.4. Tanya and Ellen’s fraction flag](image)

Ellen: The edge parts are easy—that’s just two-sixths [of a whole sheet]—but the middle part is hard.

Tanya: That’s because it’s a twenty-fourth on top of a twenty-fourth.

Ellen: I can see the twenty-fourth in the middle, but I don’t get the two little pieces on its sides.

Tanya: [Sliding over the top twenty-fourth piece] Oh, I get it. Those two side parts make a half of a twenty-fourth together, and that’s a forty-eighth.

Ellen: Okay! So the total covered on the flag is two-sixths plus one twenty-fourth plus half a twenty-fourth.

Tanya: Right! So that’s four, eight, nine-and-a half twenty-fourths. What’s that in forty-eighths?

The two girls solved the coverage problem by rearranging the pieces so that they could “see” the amounts involved. Although the calculation was informal, it amounted to \(\frac{2}{6} + \frac{1}{24} + \frac{1}{2} \times \frac{1}{24} = \frac{9}{24} = \frac{n}{48}\)—a significantly challenging task for students considered to be average achievers in mathematics.

The flag activity gave students an opportunity both to develop calculation skills and to invent situations that required more flexible strategies.

Students were positioned as mathematical doers and thinkers: “It’s good. It’s a little challenging; It’s not boring to do” [Ellen]; “A way of asking questions about the world” [Greg].

Quality pedagogy

Enactment of the flag activity highlights factors that supported productive learning:

- Task design was premised on a combination of a predetermined learning trajectory based on teacher knowledge of fractions and significant milestones.
- The teacher responded flexibly to students’ interest and learning needs. The introduction of the flag activity was a direct result of the teacher attending to the learners—learning from them—to structure responsively a more effective learning environment.
- The flag activity is a mathematically rich activity that invited exploration and conjecture while offering opportunities for personal and aesthetic expression.
- The flag setting allowed students to interact with one another, with the teacher, and with the objects in their world. The resulting conversations enabled the teacher to observe and respect the diversity in students’ fractional thinking and in their expression.
- Class discussion supported the introduction and encouragement of the use of standard fraction language and symbols.
- Students were able to create problems for themselves that were appropriate to their own levels of understanding.

These pedagogic strategies involve “not simply helping students to learn but, more fundamentally, learning from the learners” (p. 19).
**Using tools to support learners’ mathematical thinking**

As we have seen in the previous CASEs, children’s exploration of fractions can usefully involve a range of representational tools including drawings and diagrams, symbols, and manipulatives such as paper rectangles. These representations assist learners to focus on certain key features of fractions.

As discussed in chapter 5, tools can also support students’ strategic thinking and help make their solution strategies visible to others. In CASE 2, Empson (2003) noted that allowing children to choose their own tools and make their own representations to solve equal-sharing problems fosters “an interesting diversity of thinking, which can contribute to richer understanding of the mathematics of fractions” (p. 35). For example, when using part–whole representations, sixths can be drawn in three ways: by making halves and partitioning each into thirds, by making thirds and partitioning each into halves; and by making a guess about how big a sixth is and partitioning the pieces one by one. Each method supports different mathematical representations of equivalence. The first justifies the equivalence of $\frac{1}{2}$ and $\frac{3}{6}$; the second justifies the equivalence of $\frac{1}{3}$ and $\frac{2}{6}$; and the third lends itself to the idea that $\frac{1}{6}$ is equivalent to $\frac{1}{1}$.

CASE 4 examines the use of a real context, sharing cakes, in which students move from having a model ‘for’ division to a model ‘of’ division. Division of fractions is readily acknowledged as the most complex of the arithmetical operations (Ma, 1999). Fraction division can be explained as an extension of whole-number division categorisations—measurement division, partitive division, and the inverse of a Cartesian product. Division as the determination of a unit rate and division as the inverse of multiplication are two further important fraction-division interpretations (Sinicrope, Mick & Kolb, 2002). Traditionally, children have been taught division as a rule-based procedure, with little attempt to ground this procedure in a meaningful context. The researchers in this and several other studies claim that building connections with informal knowledge and encouraging students to explore multiple explanations allows students to develop a robust understanding of rational numbers.

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**CASE 4: Dividing the Cake**

*(from Sarage, 1992)*

Mathematics teaching for diverse learners:
- involves respectful exchange of ideas;
- provides opportunities for children to resolve cognitive conflict;
- provides for both planned and spontaneous/informal learning;
- utilises tools as learning supports;
- provides opportunities for students to problematise activities based in realistic contexts;
- builds on students’ prior knowledge and experiences.

**Targeted learning outcomes**

Solving problems involving division with fractions.

**Learning context**

A class discussion involving the solution of the problem "Four block cakes are to be divided into portions of three-fifths of a cake. How many portions are there?"

**Student activity**

Following exploration of the problem, the students were required to share their solution strategies with the class.

Vivian: I’ll show my diagram [going to the board to demonstrate her solution]. See, here are the four cakes (fig. 7.5). And you can see that you get six portions out of them.
Fig. 7.5. Four cakes divided into portions of $\frac{1}{5}$.

Claire: What about the left overs?

Vivian: [After a glance at her diagram] Okay, so it’s 6 and $\frac{1}{5}$.

Jonathan: Wait a minute. When I do the calculation I get 6 and $\frac{2}{3}$.

Vivian: But look at the picture. You can see that it’s $\frac{1}{5}$ left over.

Seeing the significance of the contradiction, the teacher interrupts their debate to make sure the rest of the class sees it too. Instructed to work with a partner, the students puzzle over what to do with the two pieces of cake that are left over after six people take their portions. When the class is called together again, Carole attempts her group’s explanation:

Carole: The problem asks about portions. You can say that there are 6 portions with $\frac{1}{5}$ of a cake left over or you can say that there are 6 and $\frac{1}{5}$ portions.

Sandy: Look at something else in Vivian’s diagram. She started with 4 cakes. Then she cut each cake into 5 pieces. So she had $4 \times 5 = 20$ pieces. Then she grouped those pieces by threes since 3 pieces make up a portion. So she got $20 \div 3 = 6\frac{2}{3}$ portions. She multiplied by the denominator and divided by the numerator. Like in flip and multiply!

Eleanor: I see something else in that diagram. You’ve got $\frac{1}{5}$ of a cake equal to a portion. But you can see that each cake is one portion plus another $\frac{1}{5}$ of a portion. That is, each cake is $\frac{1}{5}$ of a portion. So when you want to find out how many portions there are in 4 cakes, you can divide by the size of each portion $(4 + \frac{1}{5})$ or you can multiply by the number of portions per cake $(4 \times \frac{1}{5})$. Amazing!

Both Sandy and Eleanor are describing the meaning they now find in the usually mysterious rule for dividing by fractions.

Learner outcomes

Learning discourse of enquiry encouraged students’ struggle to resolve conflicts or confusions in their thinking. The discussion resulted in students’ productive reorganisation of the mathematical ideas into more complex levels of understanding. By allowing these students to proceed with an explanation, even when their initial answer was wrong, the teacher fostered an expectation that the mathematical authority resides within the mathematical justification, to be shared and endorsed by both teacher and students. Sharing the locus of authority meant that students in this class were free to develop confidence in their own methods and their own monitoring skills when deciding whether something made sense. Rather than trying to uncover what the teacher wanted, students were "free to focus their attention on developing justification for their methods and solutions based on the logic of mathematics” (Hiebert et al., 1997, p. 41).

Quality pedagogy

In this episode, the teacher viewed the students’ activity as meaningful—their errors were treated as building blocks to understanding. Conceptual understanding was supported by pedagogical practices that provided:

- opportunities for students to experience constructive doubt and conflict;
- opportunities for students to use familiar representations to develop, explain, and monitor their thinking;
- challenging contextual tasks that have a clear mathematics focus and purpose;
- a variety of tools to facilitate informal communication;
- opportunities for students to take the initiative and experience ownership in their learning;
- opportunities for students to engage in meaningful mathematical practices within a supportive learning community.
Task engagement and sense making

As we have seen in the previous cases, students’ sense making is a key focus of their activities. Providing students with rich tasks within an enquiry-type environment may or may not produce the desired learning activity and outcome (Henningsen & Stein, 1997). In CASE 5, we see how the first assigned task, involving rate calculations, failed to provoke appropriate solution strategies. Exposing teacher indecision as to what to do in such a situation, the case documents a successful way forward.

CASE 5: Calculating Rates
(from Smith, 1998)

Mathematics teaching for diverse learners:
• involves respectful exchange of ideas;
• provides opportunities for children to resolve cognitive conflict;
• involves sequencing of tasks and provision of appropriate challenge;
• utilises appropriate tasks with a mathematical focus—e.g., extreme examples;
• provides opportunities for students to problematise activities based in realistic contexts;
• involves explicit instructional discourse.

In this case, the teacher’s first attempt to stimulate students’ solution strategies with a rich task only served to affirm existing misconceptions. The teacher needed to reconsider how to challenge these existing misconceptions with a new task.

Targeted learning outcomes
Use of rational equations to solve problems involving rates.

Learning context
Working in a senior secondary remedial mathematics course, the teacher introduced the unit on “rational equations” (p. 750) with the following ‘Two Hands Are Better than One’ problem:

Darlene and John Edinger were looking for someone to paint their front porch. They received several estimates. The two best estimates came from Michael and Tim. Michael said that he could do the job in 8 hours. Tim told the Edingers that he could complete the job in only 6 hours. Darlene and John wanted the porch painted as quickly as possible, so they decide to hire both men. Approximately how long should it take for Michael and Tim—working together—to complete the job?

Students were required to solve this problem in groups, justifying their solutions to each other. The teacher walked around the room, monitoring their progress.

Student activity
Although the reasoning processes differed in appearance, the group solutions were all based on the misconception that the two painters would work at a rate determined by \( \frac{\text{average time}}{2} \). For example:

Group A
Michael = 8 hrs (2 people = \( \frac{1}{2} \times 8 = 4 \)); Tim = 6 hrs (2 people = \( \frac{1}{2} \times 6 = 3 \)). So, both together = \( \frac{3 + 4}{2} = 3.5 \) hrs.

Group D
8 + 6 = 14, 2 people = 7. The Edingers need the job done as quickly as possible, so they hired both boys. So you take the average of both and divide that number by 2 people. You get the time it will take 2 people to do the job, 3.5 hours.

Teacher reflections
The teacher listened in on a group discussion in which Sally put forward the following argument: “Well I know that the average is seven, and somehow you have to do this, since you are having both of them ... I know that seven is not the answer. But we have to find their time together, and I think we need to know the average to get it.” The teacher reflected to himself:
I hadn’t anticipated this approach! I don’t really see how the average could be useful. Should I say something now? Should I let them continue to pursue this conjecture? I’m really not prepared to address this misconception at the moment. This is not what I was expecting. Everyone seems to agree with Sally. Isn’t anyone going to question her conjecture? ... It looks like the whole group is buying into this idea ... Should I be the one to question it? ... What a mess!

A new problem

[After some consideration] the teacher constructed a new problem for which the ‘average time ÷ 2’ method was clearly not a sensible approach:

Suppose that Michael could complete the job in 10 hours and that Tim could complete the job in 2 hours. How long would it take the two men working together to complete the job?

Student activity

Groups initially applied the \( \text{average time} \div 2 \) algorithm to the new problem. But in some groups, the student reflections on the answer caused some questioning of this approach:

Kerri: I got three.
Hannah: I did too.
Lisa: Wait, you guys, this answer can’t be right. It only takes Tim two hours to do it alone.
Tabitha: [attempting to explain this seemingly impossible answer] Maybe Michael slows Tim down—that could happen. When I work with someone who is real slow, it happens to me.
Lisa: No. Three is right. We did something wrong.
Kerri: [agreeing with Tabitha] Maybe Tim did slow down. Why don’t the Edingers just hire Tim? Michael is too slow.

Teacher support

When the teacher returned to the whole-class discussion the students were asking good questions and were ready to move forward. They collectively confirmed their suspicion that the ‘average time ÷ 2’ algorithm was not appropriate. Realising that the students needed further guidance if they were to proceed, the teacher offered the following suggestion: “Consider each worker’s rate per hour”. The groups returned to the problem and successfully found the exact solution, \( t = \frac{1}{\frac{1}{10} + \frac{1}{2}} = \frac{30}{3} \text{ hrs} \) or \( \frac{1}{\frac{1}{10} + \frac{1}{2}} = \frac{30}{3} \text{ hrs} \) [or \( \frac{1}{\frac{1}{10} + \frac{1}{2}} = \frac{30}{3} \text{ hrs} \)].

Quality pedagogy

Factors that facilitated students’ sense making with these rate problems included:

- encouragement for students to explain their solutions and develop their own sense of accuracy;
- use of teacher questions to elicit explanations and guide students toward persuasive justification of their solutions;
- a focus on conceptual rather than procedural content. The teacher-provided information focused students on the measurable attribute of the objects in the problem (rate per hour) and the relationships among quantities;
- use of extreme examples;
- respect of student ideas and a recognition that errors can be a useful starting point for effective discussion.

Respectful exchange of ideas

Research evidence presented in chapter 5 indicated that students’ sense making involves a process of shared negotiation of meaning. In discussion-intensive learning environments, a high proportion of content development occurs through collective argumentation and group discussion. Sharing strategies and being involved in collaborative activities does not, however, necessarily result in students engaging in practices that support the development of
mathematical understanding. Shaping productive mathematical dialogue involves establishing classroom sociomathematical norms that are specific to students’ mathematical activities (Cobb et al., 1993; Yackel & Cobb, 1996). In CASE 6, classroom exchanges centred on fraction tasks illustrate how sociomathematical norms govern classroom discussions and potential opportunities for learning.

**CASE 6: Sharing biscuits**

(from Kazemi and Stipek, 2001)

Mathematics teaching for diverse learners:
• creates a space for the individual and the collective;
• provides opportunities for children to resolve cognitive conflict;
• involves the respectful exchange of ideas;
• involves explicit instructional discourse;
• provides opportunities for cognitive engagement and press for understanding.

Kazemi and Stipek (2001) observed the interactions within four teachers’ lessons. They found that, despite the outward appearance in all classrooms of a focus on understanding, opportunities for students to participate in mathematical enquiry varied across the classrooms. Their study highlights those sociomathematical norms that promote students’ engagement in conceptual mathematical thinking and conversations.

**Targeted learning outcomes**

The desired outcome of these tasks was for students to construct mathematical understanding about addition of fractions and to become skilled at communicating in mathematical language as they described and defended their differing mathematical interpretations and solutions.

**Learning context**

The study involved four teachers of grade 4 and 5, primarily low-decile classes, all teaching the same lesson on addition of fractions. The lesson involved the partitioning of brownie biscuits. The lesson plan provided one sample problem based on equivalence. Tools suggested for student use included sheets of paper with 16 pre-drawn squares.

On the surface, students appeared to be focused on understanding mathematics. However, a closer analysis of the interactions revealed differences in the way in which students were engaged in mathematical practices. In two of the four classrooms, the researchers categorised the interactions as consistently creating a high press for conceptual thinking. The other two classrooms were characterised as demanding a lower press for conceptual thinking.

The students were required to persuade each other by clear explanation and reasonable argument of their answer. In establishing rules of group participation, the teacher tried to ensure that the arguments for the errors were clearly voiced and justified.

**Mathematical argumentation**

This first vignette illustrates how students provided explanations that went beyond descriptions or summaries of the steps they used to solve the problem: they linked their problem-solving strategies to mathematical processes.

Luis: There were six crows, and we made, like, a colour dot on them ... There were four brownies, and we divided three of them into halves and the last one into sixths. One of the crows got 1/2 and 1/6.

Chris: Each crow got 1/2 and 1/6. In our second step, we had three brownies and we divided them in half. So each crow got 1/2. 1/2 plus 1/6 equals 4/6. So we have 1/2 and 1/6 and right here is 4/6. [He points to two squares: one divided into half and then 3/6 and the other into sixths. 4/6 had been shaded in each brownie; see figure 7.6.]

Luis: Just to prove that it’s the same. Then 4/6 is what they got here, plus 1/6. And 1/6 is equal to 1/6. 1/6 plus 1/6 is equal to one whole and 1/6.
It is clear from this episode that the students understood the need to demonstrate their mathematical argument for equivalence graphically as well as verbally. Kazemi and Stipek noted that the teacher invited everyone, not just the students at the board, to think about how the students had solved the problem. In the resulting discussion, the teacher required that students focus on the mathematical concept of equivalence and its relation to the process of adding fractional parts; it was not enough that students commented on the clarity of the drawings or how they were shaded.

To illustrate the effect of contrasting expectations, Kazemi and Stipek provide an episode from another classroom in which students engaged in the same social practices of describing their thinking but merely summarised the steps they took to solve a problem. The episode follows on from Raymond’s description of his solution for dividing 12 brownies among eight people. The teacher, Ms. Andrew, had drawn 12 squares on the chalkboard. Raymond divided four of the brownies in half.

Ms. A: Okay, now would you like to explain to us what ...
R: Each one gets one, and I give them a half.
Ms. A: So each person got how much?
R: One and 1/2.
Ms. A: 1/2?
R: No, one and 1/2.
Ms. A: So you’re saying that each one gets one and 1/2. Does that make sense? [Chorus of “yeahs” from students; the teacher moves on to another problem.]

Ms. Andrew did not ask students to justify why they chose a particular partitioning strategy. A commonly observed practice in Ms. Andrew’s class was for students to indicate with a show of hands or by calling out ‘yes’ or ‘no’, their responses to questions such as “How many people agree?” “Does this make sense?” or “Do you think that was a good answer?” These general responses, while on the surface indicating participation, revealed limited information about students’ thinking or their understanding of the mathematical concepts involved.

**Understanding relationships**

Sharing solution strategies enables students to reflect on relationships within mathematics. The following vignette illustrates how one of the teachers in Kazemi and Stipek’s study supported her students’ examination of the mathematical similarities and differences among multiple strategies.

After Michelle and Sally had described their strategy and solution for a fair-sharing problem (see figure 7.7), the teacher turned to the class:

Ms. M: Does anyone have any question about how they proceeded through the problem? ... What did they use or do that was different than what you might have done?

Jeff: They used steps.
Ms. M: Right, they divided it into steps. But there were some steps that I haven’t seen anyone else use in the classroom yet.
Carl: They added how many brownies there were altogether.
Ms. M: Okay, so they used …
Jan: They divided into six, and there was one left over, and then they figured how they were going to divide that equally so that every crow gets a fair share.
Ms. M: Exactly, and that was very observant of you to see that. As I walked around yesterday, this is the only pair that used a division algorithm to determine that there was a whole brownie and a piece left over. So they did it in two different ways.

Fig. 7.7. Sharing 4 brownies among 6 crows and then 3 more brownies among 6 crows

In this exchange, the teacher asked her students to reflect on what was unique about a particular group’s solution strategy. Students’ responses included both organisational (“they divided it into steps”) and mathematical (“they divided it into sixths”) aspects. The researchers noted that the focus on mathematical differences among shared strategies supported students’ formation of mathematical connections between various solution paths.

In contrast, in the ‘low press for understanding’ classrooms, discussions focused on non-mathematical aspects of shared strategies. The sharing of strategies “looked like a string of presentations, each one followed by applause and praise” (p. 72). Links, if they were made, consisted of non-mathematical aspects. For example, in Ms. Andrew’s class, a pair of students reported that a solution strategy involved cutting the brownies and distributing the pieces to each individual. Another student reported drawing lines from the fractional parts of the brownies to the individuals who received them. Although both solutions used partitioning strategies, they were accepted as mathematically different, based on the way they handed out the pieces.

Building on errors

In classrooms where student participation and contribution is valued, student errors provide entry points for further mathematical discussion. Errors, or inadequate or partial solutions, provide opportunities to reconceptualise a problem, explore contradictions in a solution approach, or try out alternative strategies.

In the following episode, Ms. Carter organised a discussion based on conflicting student responses to the problem. By encouraging the whole class to think about a range of possible solutions, she created an opportunity for all students to engage in mathematical analysis. Following on from an earlier class presentation in which Sarah and Jasmine offer a diagrammatic solution to the problem of dividing nine brownies equally among eight people (fig. 7.8), Ms. Carter is involved in the following classroom exchange:

Ms. C: Do you want to write that down at the top [of the OHP] so I can see what you did? [Jasmine writes \( \frac{1}{2} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \)]

Ms. C: Okay, so that’s what you did. So how much was that in all?

J: It equals \( \frac{1}{6} \) or \( \frac{4}{6} \)

Ms. C: So she says it can equal \( 6 \) [sic] and \( \frac{4}{6} \)?

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In this episode, the teacher could have stepped in and pointed out why \( \frac{6}{8} \) and \( \frac{11}{8} \) are not equal. Instead, her response was to encourage her students to explore the error by explaining why \( \frac{6}{8} \) and \( \frac{11}{8} \) can’t be equal.

Kazemi and Stipek also noted that Ms. Carter frequently used her observations of inadequate solutions during group work to plan whole-class instruction. In contrast, although incomplete or inadequate solutions were accepted as a normal part of learning in the other two classrooms, they were more likely to be ignored by the teacher or passed over until an adequate solution was offered or corrected by the teachers themselves.

**Collaborative argumentation**

In classes where there was a high press for understanding, collaborative work was accompanied by an expectation that each student was accountable for thinking through the mathematics involved in a problem. There was an expectation that consensus should be reached through mathematical argumentation. Teachers in both classrooms provided guidelines for small-group participation and reinforced these expectations. When Ms. Martin began the work on the fair-share problems, she made the following statement regarding individual accountability:

**Teacher:** Everyone in your group, whether it’s just the two of you, or the three of you, everyone in your group needs to understand the process that you all were supposed to go through together. Because when you make a presentation, you don’t know whether or not you are going to be asked a question. So you don’t know if you’re going to be asked by me or by your classmates. So you need to make sure that each person understands each part of the process you went through.

When working in groups, these expectations appeared to have been understood in terms of distribution of labour and contribution. The following episode is typical of an interactive discussion:

**Keisha:** See, this is how I explained it. [Reads.] “What we did is we took three brownies and cut them into half because three plus three equals six. And there are six crows.”

**Mark:** This is what I put so far. [Reads.] “We knew that there were three more brownies, and we divided each one in half. One brownie had two halves, and another brownie had two halves, and another one had two halves.”

**Keisha:** Just write, “And all three had two halves.” And all …

**Mark:** [Starts writing.] “And all three …” I don’t have to write.

**Keisha:** Okay, and each crow got \( \frac{1}{2} \), and … just write: “\( 3 \times 3 \) is 6” so each crow got \( \frac{1}{2} \) of the brownie.

**Mark:** Yeah, but these are halves. [Counts halves in each brownie.] 2, 2.

**Keisha:** Yeah, I know. 1, 2, 3. [Counts brownies.] Cut them in half, 1, 2, 3, 4, 5, 6. [Counts halves.] There were six crows. Each crow got \( \frac{1}{2} \).

**Mark:** [Writes “Each crow got one-half.”]

Having written about one of the steps in the problem, Mark and Keisha proceeded to evaluate and expand their written explanations. We see in the above interaction how Mark indicated that he did not appear to understand how \( 3 \times 3 \times 6 \) applied. Keisha’s explanation that she was referring not to the halves themselves,
but that the number of halves corresponded to the number of crows appears to have facilitated a mutual understanding.

In contrast to the high-press classrooms, the teachers in the low-press classrooms only gave general instructions (such as “work with a partner” or “remember to work together”) to support collaboration. The researchers noted frequent occurrences of unequal distribution of work within groups. Students who were unclear about what to do often withdrew and allowed another student to take over. In the following example, Ellen was excluded from participation. Without an opportunity to think about the problem, her role was one of ‘listen and agree’:

Lisa: We need five brownies. So see, 1, 2, 3, 4. So we cut these into half. So 1… 8. They get a half each. And then there’s one more cookie and eight people, so we just cut into eightths, and it’ll be even for everybody.

Ellen: Wow, you did that fast. I didn’t even do anything.

Lisa: I knew there’s 5, and I knew 4; 2 times 4 is 8.

Ellen: Oh, I get it.

Learner outcomes
Students in classrooms with established social and sociomathematical norms for mathematical thinking were more likely to be observed engaging in mathematical discussion that was conceptual rather than procedural in nature.

Quality pedagogy
The difference between the high- and low-press exchanges illustrates that, to support conceptual thinking, teaching needs to go beyond superficial practices based on the discussion and sharing of strategies. Kazemi and Stipek argue that pedagogical practices need to be based on the following sociomathematical norms that work together to establish expectations about what constitutes mathematical thinking:

• An explanation consists of a mathematical argument, not simply a procedural description or summary.
• Mathematical thinking involves understanding relations among multiple solutions.
• Errors provide opportunities to reconceptualise a problem, explore contradiction in solutions, or pursue alternative strategies.
• Collaborative work involves individual accountability and reaching consensus through mathematical argumentation.

Belonging to a community of learners
Engaging students in what Wood, Cobb, & Yackel (1991) call “genuine conversations” about mathematics means that teachers take students’ ideas seriously in their attempts to support students’ understanding. CASE 7 provides a window into a lesson that involves ratios. In addition to featuring the use of multiple connections between percentage, fractions, and measurement, the detailed exchange between the teacher and student illuminates the sense of community and ethic of care that pervade this classroom.

CASE 7: Mixing Drinks
(from Sherin, Mendez, and Louis, 2004)

Mathematics teaching for diverse learners:
• demands an ethic of care;
• involves the respectful exchange of ideas;
• creates a space for the individual and the collective;
• provides opportunities for students to resolve cognitive conflict;
• involves sequencing of tasks and provision of appropriate challenge;
• provides opportunities for students to problematise activities based in realistic contexts;
involves explicit instructional discourse.

This case is taken from a wider curriculum reform programme, Fostering a Community of Learners (FCL; Brown & Campione, 1986). The goal in the mathematics classroom was the building of a discourse community that focused on students’ explanations and discussion of their ideas. In the following example, we see how the FCL principles of activity, reflection, collaboration, and community are realised in David’s classroom.

Targeted learning outcomes
Solving ratio problems.

Learning context
The students in this study were from a middle-school classroom involved in an extended reform programme intervention in conjunction with researchers at Stanford University. Data were collected through videotapes of instruction and through discussions of these videotapes with the teacher. The lesson reported on in this case occurred in the second year of the study. Students were given a series of drink mix problems—one of these was as follows:

Juice Mix A contains 2 cups of concentrate and 3 cups of cold water. Assuming that each camper will get ½ cup of juice, how much concentrate and how much water are needed to make juice for 240 campers?

While the students worked on the problem in groups, David, the teacher, circulated through the class.

Student and teacher activity
As David approached Antoine’s group, Antoine called him over for help on the last problem. David asked Antoine what he had done so far.

Antoine: OK. You do 3 out of 5. Three divided by 5 is 60%, times 240 equals 144, divided by 2 is 72. I’ve got the answer. I’ve got skills, boy. Yeah.

Teacher: Can you explain what you just did, what that means?

Antoine: Yeah, yeah.

Teacher: What’s the 3 out of 5 part?

Antoine: Three out of 5 is the number, the number, the cups of water divided by all of the cups put together. And then it equals 60%. And then, times 240 is the number of campers.

Antoine had recognised that for every five cups of juice, three of the cups were water. And it seemed that because 3/5 = 0.6, Antoine concluded that 60% of the juice mix must be water, no matter how many cups of juice. He then took the total number of campers, 240, and multiplied that by 0.6. Because each camper gets only half a cup of juice, Antoine divided 144 by 2 to get 72 cups of water in the total mix. Later on in the discussion, it became clear that Antoine was unsure of this last part of his calculation.

The report of the teacher’s reflections of this episode noted that he was unclear as to why Antoine would multiply 0.6 by 240—the number of campers. Why did Antoine need to know how many 60% of the campers would be? David’s own solution involved first calculating the total numbers of cups of juice that were needed, 120 cups and then because 60% of the juice was water, calculating 60% of 120 to conclude that there 72 cups of water. David asked Antoine to elaborate his solution method.

Teacher: So that tells you what, if you do 60% times 240?

Antoine: It tells you how many cups, wait. Times 240. Tells you how many cups are needed.

Teacher: That tells you 60% of the campers.

Antoine: No, tells you 60% of that juice stuff.

Robert: Tells you that 60% of the mix is concentrate.

Antoine: No, it tells you that 60% of water is in the mix altogether.

Robert: Yeah.

Teacher: All right, all right, you’ve got skills. Let’s go.

Antoine: I know I’ve got skills, you ain’t got to tell me. Then this is the part I messed up at. The number of campers, you get. You have to times it, and then what do you get?

In the above episode, we see that Antoine’s explanation initially involves ‘how many cups’ are needed. In response to the teacher probe Antoine becomes a little more precise, explaining that it tells him ‘60% of
that juice stuff’. Robert, another group member, interjects with an incorrect statement, saying that it tells you that ‘60% of the mix is concentrate’. However, in trying to respond to Robert, Antoine is finally able to explain that 60% of 240 tells him how much ‘water is in the mix altogether’.

**Quality pedagogy**

The researchers analysed factors that facilitated students’ success with the ratio problems in terms of four principles of learning: activity, reflection, collaboration, and community.

- Antoine is clearly an active participant in the discussion. He seeks teacher assistance and enthusiastically reviews his solution method with the teacher.
- The interaction supports Antoine to “reflectively turn around on [his] own thought and action and analyse how and why [his] own thinking achieved certain ends or failed to achieve others” (Shulman, 1995, p. 12, cited in Sherin et al., 2004). In reviewing his solution, Antoine clearly wants to resolve his uncertainty as to why he decided to divide 144 by 2.
- Collaboration involves teacher–student and student–student interactions. In the second episode, Robert and Antoine scaffold and support each other’s learning in ways that supplement each other’s knowledge.
- Sherin, Mendez, and Louis claim that the ‘community’ principle is more clearly evident in the videotape. From the transcripts, it is apparent that the teacher and Antoine have an established routine that supports effective communication. Teacher questioning occurs in an environment in which Antoine feels safe to respond. Antoine seeks teacher help, knowing that it is appropriate to question his own solution despite the fact that he already has the answer. The banter between Antoine and the teacher provides evidence of a culture that affords opportunities for students to share their understandings and to know that their opinions are valued.

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**Teacher knowledge: Forms of ‘knowing’ fractions**

In all of the CASEs, the centrality of teacher knowledge is evident. Reiterating discussions from earlier chapters, what teachers do is very dependent on what they understand about the teaching and learning of mathematics. For fractions, in particular, knowing different models and various approaches to the teaching of fractions places high demands on teachers’ mathematical and pedagogical content knowledge.

Unfortunately, numerous studies point to shortcomings in teachers’ understanding of rational numbers (e.g., Domoney, 2001). In a seminal study comparing US and Chinese teachers’ mathematical knowledge, Ma (1999) demonstrated how US teachers more readily situated fraction problems in real-world contexts. However, this apparent familiarity with and link to everyday experience appeared to be superficial. Ma found substantial differences in knowledge when teachers were asked to perform division of fractions or to generate representations of fractions. For example, only one of the 23 US teachers in the study generated a conceptually correct representation for the meaning of the equation $1\frac{3}{4} ÷ \frac{1}{2}$. This compares with 65 of the 72 Chinese teachers. Among the 23 US teachers, 6 could not create a story to match the calculation and 16 provided stories that contained misconceptions. Twelve of the misconceptions involved confusing division by $\frac{1}{2}$ with division by 2 or multiplication by $\frac{1}{2}$. For example: *Jose has one and three-fourths boxes of crayons and he wants to divide them between two people or divide the crayons in half, and then, first we could do it with crayons and maybe write it on the board or have them do it in numbers.* Ma cautioned that although US teachers reported the frequent use of real contexts, the ‘real world’ cannot produce the mathematical content by itself. She claims that “without a solid knowledge of what to present, no matter how rich one’s knowledge of students’ lives, no matter how much one is motivated to connect mathematics with students lives, [this is without benefit] if one still cannot produce a conceptually correct representation” (p. 82).
CASE 8: Representation of Division by Fractions
(from Ma, 1999)

All of the Chinese teachers successfully computed \( \frac{11}{2} \div \frac{1}{2} \) and 65 of the 72 created a total of more than 80 story problems representing the meaning of division by a fraction. The Chinese teachers represented the concept using three different models of division: measurement (quotitive), partitive (sharing), and product and factors. For example, \( \frac{11}{2} \div \frac{1}{2} \) might represent:

- \( \frac{11}{2} \) metres \( \div \) metre \( \times \) \( \frac{1}{2} \) (quotitive model)
- \( \frac{11}{2} \) metres \( \div \) \( \times \) \( \frac{1}{2} \) metres (partitive model)
- \( \frac{11}{2} \) square metre \( \div \) metre \( \times \) \( \frac{1}{2} \) metres (product and factors)

corresponding to the problems:

- How many \( \frac{1}{2} \) m lengths of timber are there in \( \frac{11}{2} \) m of timber?
- If half a length of timber is \( \frac{11}{2} \) m, how long is the whole piece of timber?
- If one side of a \( \frac{11}{2} \) square metre rectangle is \( \frac{1}{2} \) m, how long is the other side?

In their discussions of the meaning of division by fractions, the Chinese teachers mentioned several concepts that they considered related to the topic. These are represented in the diagram:

![Fig. 7.9. Connected knowledge for understanding the meaning of division](image)

Their view of connected knowledge translated into a forward trajectory of learning. Work on division by fractions was also valued for the role in intensifying concepts of rational number already encountered by the students. The Chinese teachers expressed the view that students may “gain new insight through reviewing old ones [concepts]. The current learning is supported by, but also deepens, the previous learning” (p. 77).

A more recent comparative study of US and Chinese teachers (Shuhua, Kulm, & Wu, 2004) also notes the different system demands on teachers’ pedagogical content knowledge. The researchers express concern about the pedagogical approaches in the US system that indicate a “lack of connection between manipulative and abstract thinking, and between understanding and procedural development” (p. 170).

As we have seen earlier, teacher knowledge is also a significant factor in the interpretation of students’ thinking and solution strategies. For example, in assessing students’ understanding of fractions, sound pedagogical content knowledge is needed to determine the effectiveness of different tasks.
CASE 9: Knowledge of Students Solving Fractions
(from Grossman, Schoenfeld, and Lee, 2005)

Which of the following tasks would best assess whether a student can correctly compare fractions?

- Write these fractions in order of size, from smallest to largest: $\frac{5}{8}, \frac{1}{4}, \frac{11}{16}$
- Write these fractions in order of size, from smallest to largest: $\frac{5}{8}, \frac{3}{4}, \frac{1}{16}$
- Write these fractions in order of size, from smallest to largest: $\frac{5}{8}, \frac{3}{4}, \frac{11}{16}$

Can you explain why the two tasks you did not select are not good assessments of students’ understanding of fractions?

To do this, teachers need to know the relevant mathematics in a deep and connected way (Ma, 1999). Teachers need more than just the ability to solve the problem. They also need to know about the ways in which students might solve the problem and the reasoning that they may or may not use. For example, there are at least two ways to solve the problem: converting the fractions to decimals and comparing them or reasoning through the task by comparing the fractions themselves. Of the three fractions in the first set, only $\frac{1}{4}$ is less than $\frac{1}{2}$. So $\frac{1}{4}$ is the smallest. And then because $\frac{5}{8} = \frac{10}{16}$, and $\frac{10}{16}$ is less than $\frac{11}{16}$, $\frac{5}{8}$ is less than $\frac{11}{16}$. Thus the order is $\frac{1}{4}$, $\frac{5}{8}$, $\frac{11}{16}$. But the real issue for the teacher is how their own students will solve this problem.

Research shows that many students will focus only on the number of pieces, not their relative size. For example, given the first set of three fractions, students will think, “$\frac{1}{4}$, has only one piece, so it’s the smallest; $\frac{5}{8}$ has five pieces, so it’s in the middle; and $\frac{11}{16}$ has eleven pieces, so it’s the largest.” Unfortunately this incorrect reasoning produces the right answer. Another common misconception held by students is that “the smaller the pieces, the smaller the fraction.” Because sixteenths are smaller than eighths and eighths are smaller than fourths, students using this reasoning may arrive at the correct answer, given the second set of three fractions. In the case of the third set of three fractions, however, students who use either of these forms of incorrect reasoning will get the wrong answer—and their wrong answer will suggest why they got it wrong.

With fractions, as with other areas of mathematics, teachers need to distinguish what the student understands as opposed to what the student can do (Pearn & Stephens, 2004). The teacher may need to listen across multiple tasks in order to determine how a student is thinking. The following response by Madison, a student in Mitchell and Clarke’s (2004) research, illustrates the value of multiple tasks in the diagnostic setting. When Madison was asked to respond to an estimation task involving adding pairs of fractions near 1 and near $\frac{1}{2}$, her estimate for $\frac{7}{8} + \frac{12}{13}$ was “two”—the correct answer. According to the researchers, her reasoning appeared modest but faultless: “I just guessed. That’s seven bits of eight. Twelve bits of thirteen.” Based on the assumption that Madison had used a part–whole approach, her follow-up comment seemed strange and unrelated to her answer, “And I just added eight and thirteen”. Madison was then asked to estimate the answer to $\frac{7}{8} + \frac{1}{2}$, to see if she could use half as a benchmark. Her answer “twelve” was accompanied with the following reasoning: “But eight and twelve are twenty and the three and the five are covering a bit of it and so I took it away.” It appeared that adding the numerators and taking away that from the sum of the denominators was her procedure. Applying this whole-number procedure to the previous question, it is clear how she arrived at the answer of two; $8 + 13$, which is what she said she did, is $21$. $21 - (7 + 12)$ is 2.

These examples of teachers’ knowledge about fractions were sourced from research studies that interviewed teachers. They serve to reinforce the evidence that we have presented throughout the synthesis, which clearly illustrates the critical relationship between teacher knowledge enacted in the learning environment and learner outcomes. Specifically, teachers who have strong pedagogical content knowledge and ‘connected’ knowledge of mathematics, and who also display an awareness of the need to ‘connect’ with learners’ understandings of mathematics, are most likely to occasion quality learning opportunities for diverse learners.
**Part–whole pedagogy**

Individually, these CASEs illuminate those aspects of effective pedagogy for diverse learners that have been discussed in earlier chapters. As a whole, they represent more than the parts of the story. They illustrate the truth that effective pedagogy cannot be captured by a list of ticks in boxes. Teaching is socially situated and contextually bound. What works well with one group of students may not work so well with another, and what works well with one mathematical topic may not work so well with another. The CASEs serve to illustrate how students and teachers are collectively learners, constructing meanings that contribute to the overall success of each learning experience. In each CASE, there is a commitment to the fundamental principle proffered in chapter 2, and that is that all students, irrespective of age, have the capacity to become powerful learners of mathematics.

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**References**


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Appendix 1: Locating and Assembling BES Data

Using the ‘health-of-the-system’ approach, we sought to examine the various factors implicated in the creation of an effective learning community. We investigated a number of measures that fell naturally from the ‘what’, ‘why’, ‘how’, and ‘under what conditions’ questions concerning pedagogical approaches that facilitate learning for all students. The task was a considerable one, involving information management, the engagement of advisory and audit groups, and the seeking of contributions from the education community in general and the mathematics education community in particular. This level of engagement ensured that the Best Evidence Synthesis would be inclusive of views from across the community.

Our initial search strategy required us to pay attention to different contexts, different communities, and multiple ways of thinking and working. With this in mind, we undertook a literature search that crossed national and international boundaries. We used a range of search engines as well as personal networks to help us find academic journals, theses, projects, and other scholarly work with a focus on mathematics in New Zealand schools and centres, and by selected authors worldwide. We searched both print indices and electronic indices, endeavouring to make our search as broad as possible within the limits of manageability. This search took into account relevant publications from the general education literature and from the literature that relates to specialist areas of education. The search covered:

- key mathematics education literature including all major mathematics education journals (e.g., *Journal for Research in Mathematics Education*, *Educational Studies in Mathematics*, *Journal of Mathematics Teacher Education*, *For the Learning of Mathematics*, *The Journal of Mathematical Behaviour*), international conference proceedings (e.g., PME, ICME), Mathematics Research Group of Australasia publications, and international handbooks of mathematics education (e.g., Bishop et al., 2003);
- relevant New Zealand-based studies, reports, and thesis databases, supported by input from the professional community and the Ministry of Education;
- education journals (e.g., *American Educational Research Journal*, *British Educational Research Journal*, *Cognition and Instruction*, *The Elementary School Journal*, *Learning and Instruction*, etc.) and New Zealand work (e.g., SAMEpapers, SET, NZJES);
- specialist journals and projects, especially those located within the wider education field (e.g., *New Zealand Research in Early Childhood Education*, *Journal of Learning Disabilities*);
- landmark international studies including TIMSS, PISA, the UK Leverhulme projects.

This search strategy led us to a large body of literature that had something to say about facilitating mathematics learning: the total number of sourced references was just over 1100.

Table 1 categorises these references by source:

<table>
<thead>
<tr>
<th>Source of data</th>
<th>Relative frequency (n ~1100)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mathematics education journals</td>
<td>24%</td>
</tr>
<tr>
<td>Mathematics education reports, books, handbooks</td>
<td>16%</td>
</tr>
<tr>
<td>Mathematics education conference proceedings</td>
<td>15%</td>
</tr>
<tr>
<td>Theses and projects</td>
<td>6%</td>
</tr>
<tr>
<td>General education reports, books, handbooks</td>
<td>10%</td>
</tr>
<tr>
<td>General education journals, reports, and proceedings</td>
<td>19%</td>
</tr>
<tr>
<td>Specialist journals</td>
<td>10%</td>
</tr>
</tbody>
</table>
All entries were stored and categorised using EndNote. To assist in the initial synthesis, we distinguished between ‘research’ and ‘discussion document’, and categorised entries according to (a) our ‘diversity’ descriptors (e.g., ethnicity, gender, socioeconomic), (b) centre/school level, and (c) country-of-origin of the data.

These categories and sub-divisions served as a useful starting point for overviewing the literature and allowed us to foreground our fundamental intent to be responsive to diversity. In addition, by classifying entries according to sector and country of origin, we gave ourselves a constant reminder of the need to be inclusive of all perspectives and interests. This inclusiveness gave us a body of literature comprising diverse frameworks and eclectic methodological and analytic approaches.

**Selecting the evidence**

Given the complexity of the teaching and learning process, it is not an easy matter to link specific outcomes with specific pedagogical approaches. In our first pass through the literature, we noted that studies could claim that student achievement was influenced by pedagogical practice much more readily than they could explain how that practice affected student achievement. Many studies offered detailed explanations of student outcomes yet failed to draw conclusive evidence about how those outcomes related to specific teaching practices. Others provided detailed explanations of pedagogical practice yet made unsubstantiated claims about, or provided only inferential evidence for, how those practices connected with student outcomes.

Granted, we were not looking for linear explanations. As Sfard (2005) points out, the complexity of the teaching–learning relationship “precludes the possibility of identifying clear-cut cause–effect relationships” (p. 407). What we were searching for were studies that were able “to offer a developing picture of what it looks like for a teacher’s practice to cultivate student [proficiency]” (Blanton & Kaput, 2005, p. 440). We were searching for studies that offered a “detailed look at how [teachers’] actions played out in the classroom and how students were involved in this” (Blanton & Kaput, 2005, p. 435) and the sorts of mathematical proficiency that resulted. Specifically, we were seeking studies that offered not just detailed descriptions of pedagogy and outcomes but rigorous explanation for close associations between pedagogical practice and particular outcomes.

Paying attention to diverse forms of research evidence required our serious consideration of the literature relating to disparate factors from different sectors and representative of different time periods. Luke and Hogan (in press) note that what is distinctive about the approach undertaken in the New Zealand Best Evidence programme “is its willingness to consider all forms of research evidence regardless of methodological paradigms and ideological rectitude, and its concern in finding contextually effective appropriate and locally powerful examples of ‘what works’... with particular populations, in particular settings, to particular educational ends” (p. 5). We have included many different kinds of evidence that take into account human volition, programme variability, cultural diversity, and multiple perspectives. Each form of evidence, characterised by its own way of looking at the world, has led to different kinds of truth claims and different ways of investigating the truth. Our pluralist stance left us free to consider the relative strengths and weaknesses of different methodological approaches.

A fundamental challenge for this BES has been to demonstrate a basis for knowledge claims. We are absolutely aware that, like data selection, assessment of evidential claims from secondary sources is a highly perspectival activity. “Even those gazing down a microscope are as capable of finding what they expect to find, or want to find, as anyone else” (Davies, 2003). In response to this challenge, studies have been reported in a way that will make the original evidence as transparent as possible. Informed by the Guidelines for Generating a Best Evidence Synthesis Iteration 2004, we included studies that:

- provided a description of the context, the sample, and the data;
• offered details about the particular pedagogy and the specific outcomes;
• connected research to relevant literature and theory;
• used methods that allow investigation of the pedagogy–outcome link;
• yielded findings that illuminated what did or did not work.

The Guidelines for Generating a Best Evidence Synthesis Iteration allowed us to deal not only with a diversity of research topics, approaches, and methods, but also to capture differences in the context, practices, and ways of thinking of researchers. The method employed in this BES for evaluating validity required us to look at the ways different pieces of data meshed together and to determine the plausibility, coherence, and trustworthiness of the interpretation offered.

Assessments about the quality of research depend to a large extent on the nature of the knowledge claims made and the degree of explanatory coherence between those claims and the evidence provided. What we were looking for was the explanatory power of the stated pedagogy–outcome link. When assessing the nature and strength of the causal relations between pedagogical approaches and learning outcomes, we were guided by Maxwell’s (2004) categorisations of two types of explanations of causality. The first type, the regularity view of causation, is based on observed regularities across a number of cases. The second type, process-oriented explanations, sees “causality as fundamentally referring to the actual causal mechanisms and processes that are involved in particular events and situations” (p. 4). Cobb argues (2006, personal communication) that regularity explanations are particularly useful for policy makers, while process-oriented explanations are relevant to teachers, who are concerned with “the mechanism through which and the conditions under which that causal relationship holds” (Shadish, Cook, & Campbell, 2002, p. 9, cited in Maxwell, 2004, p. 4). Attending to both types of explanation of causality meant including both large-scale and single-case studies.

In many instances, we have found it useful to present a single case—a learner or teacher, a classroom, or a school—in the form of a vignette to exemplify the relations between learning processes and the means by which they are supported.

**Research sources in this BES report**

This BES report contains approximately 660 references. Included amongst these are research reports of empirical studies, ranging from very small, single-site settings (e.g., Hunter, 2002) to large-scale longitudinal studies (e.g., Balfanz, Maclver, and Byrnes, 2006). Some of the larger studies have multiple references because they include different papers/conference proceedings/book chapters or because they embrace work authored by different researchers (e.g., the New Zealand Numeracy Development Project). In addition, the references include reports containing educational statistics and policy, theoretical writings, and commentaries and reviews on multiple research findings (e.g., van Tassel-Baska, 1997).

The Guidelines for Generating a Best Evidence Synthesis Iteration point to the importance of drawing on New Zealand research in order to illuminate what works in the New Zealand context. However, despite an exhaustive search for New Zealand work, it is apparent (see chapter 8 for further discussion) that the strengths and foci of New Zealand research are not evenly distributed. In some areas—for example, early years education—there are relatively few New Zealand (or Australian) researchers working with a specific focus on mathematics education (Walshaw & Anthony, 2004). Table 2 shows the country of origin of the literature included in this BES. The numbers reflect New Zealand’s relatively new positioning within the international mathematics education research community.
Table 2: Database composition according to country

<table>
<thead>
<tr>
<th>Country</th>
<th>Relative Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>New Zealand</td>
<td>27%</td>
</tr>
<tr>
<td>Australia</td>
<td>17%</td>
</tr>
<tr>
<td>United Kingdom</td>
<td>11%</td>
</tr>
<tr>
<td>United States</td>
<td>49%</td>
</tr>
<tr>
<td>Other (e.g., Africa, Netherlands, Spain)</td>
<td>6%</td>
</tr>
</tbody>
</table>

Table 3 shows the proportion of the items included in the BES (both empirical studies and commentaries) that relates to each of the different sectors. Publications relating specifically to intermediate schools have been classified with the literature on primary schools.

Table 3: Database composition according to sector

<table>
<thead>
<tr>
<th>Sector</th>
<th>Relative Frequency (n=520)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Preschool</td>
<td>18%</td>
</tr>
<tr>
<td>Primary school</td>
<td>48%</td>
</tr>
<tr>
<td>Secondary school</td>
<td>21%</td>
</tr>
<tr>
<td>Teacher education</td>
<td>13%</td>
</tr>
</tbody>
</table>

**Synthesising the data**

Our conceptual framework, outlined in chapter 2, offered a way of structuring the data. Within the community of practice frame in and beyond the classroom, we identified the following components: (a) the organisation of activities and the associated norms of participation, (b) discourse, particularly norms of mathematical argumentation, (c) the instructional tasks, and (d) the tools and resources that learners use. We began the iterative chapter-structuring process by outlining a number of key areas. These included mathematical thinking and identities, scaffolding and co-construction, tasks, activities, assessment, educational leadership, home-school/centre links, and wider school communities. Each of these served as a starting point for our exploration and was found, in the course of the investigation, to be a useful initial category for addressing questions of equity and proficiency in relation to effective mathematics teaching.

In time, we organised these categories more cohesively into groups. What we endeavoured to do was organise multiple elements, types, and levels and varying temporal conditions in line with the critical dimensions of a community of practice and the guiding principles established in chapter 2. The content of the subsequent chapters is shaped according to these dimensions and principles. Chapter 3 focuses on all three dimensions in a search for understanding of how pedagogy influences early years outcomes. Chapters 4 and 6 explore interrelationships that are centred on the joint enterprise of developing mathematical proficiency for all learners. Chapter 5 explores the role of mathematical tasks and the part that they play in enhancing students’ learning.

Reminding ourselves and readers that this BES synthesis is a product of currently accessible research, we concur with Atkinson’s (2000) view that “the purpose of educational research is surely not merely to provide ‘answers’ to the problems of the next decade or so, but to continue to inform discussion, among practitioners, researchers and policy-makers, about the nature, purpose and content of the educational enterprise” (p. 328). Rather than offering broad answers that promise much and achieve little, it is our hope that the structure we have used will foster understanding, reflection, and action concerning the characteristics of effective pedagogical approaches in mathematics.
References


Appendix 2: URLs of citations

The following 22 papers/articles/chapters/books are suggested as potentially useful sources for teachers to engage more deeply with the range of issues raised in this best evidence synthesis iteration. Readers are encouraged to source and read them. Several are available online; the others can be sourced through libraries.

The full citations are hyperlinked in the online PDF. For the convenience of those using a hard copy of the text, the URLs are listed below.

**Carpenter, Thomas P ; Franke, Megan L ; Jacobs, Victoria R**
A longitudinal study of invention and understanding in children’s multidigit addition and subtraction

**Clarke, Barbara ; Clarke, Doug**
Mathematics teaching in Grades K-2: painting a picture of challenging supportive, and effective classrooms

**Cobb, Paul ; Boufi, Ada ; McClain, Kay ; Whitenack, Jor**
Reflective discourse and collective reflection

**Empson, Susan B**
Low performing students and teaching fractions for understanding: An interactions analysis

**Fraivillig, Judith L ; Murphy, Laren A ; Fuson, Karen C**
Advancing children’s mathematical thinking in everyday mathematics classrooms

**Gifford, Sue**
A new mathematics pedagogy for the early years: in search of principles for practice

**Goos, Merrilyn**
Learning mathematics a classroom community of inquiry

**Houssart, Jenny**
Simplification and repetition of mathematical tasks: a recipe for success or failure?

**Irwin, Kathie ; Woodward, J (paper available online)**
A snapshot of the discourse used in mathematics where students are mostly Pasifika (a case study in two classrooms)

**Kazemi, Elham ; Stipek, Deborah**
Promoting conceptual thinking in four upper-elementary mathematics classrooms

**Latu, Vilami (paper available online)**
Language factors that affect mathematics teaching and learning of Pasifika students

**O’Connor, Mary Catherine**
“Can any fraction be turned into a decimal?” A case study of the mathematical group discussion

**Rietveld, Christine M.**
Classroom learning experiences of mathematics by new entrant children with Down syndrome

**Savell, Jan ; Anthony, Glenda Joy**
Crossing the home-school boundary in mathematics
Sheldon, Steven B ; Epstein, Joyce L
Involvement counts: family and community partnerships and mathematics achievement
http://nzcer.org.nz/BES.php?id=BES012

Smith, Margaret Schwan Smith ; Henningsen, Marjorie A
Implementing standards-based mathematics instruction: a casebook for professional development

Steinberg, Ruth M ; Empson, Susan B ; Carpenter, Thomas P
Inquiry into children’s mathematical thinking as a means to teacher change
http://nzcer.org.nz/BES.php?id=BES014

Watson, Anne ; De Geest, Els
Principled teaching for deep progress: Improving mathematical learning beyond methods and material
http://nzcer.org.nz/BES.php?id=BES015

Wood, Terry (paper available online)
What does it mean to teach mathematics differently?
http://nzcer.org.nz/BES.php?id=BES016

Yackel, Erna ; Cobb, Paul
Sociomathematical norms, argumentation, and autonomy in mathematics
http://nzcer.org.nz/BES.php?id=BES017

Young-Loveridge, Jenny (paper available online)
Students views about mathematics learning: a case study of one school involved in Great Expectations Project
http://nzcer.org.nz/BES.php?id=BES018

Zevenbergen, R
The construction of a mathematical habitus: implications of ability grouping in the middle years
http://nzcer.org.nz/BES.php?id=BES019
Appendix 3: Glossary

The page reference for the first and/or most important occurrence of the term is given in brackets.

Cognitive engagement (p. 2). The state of being engaged in thinking
Community of Practice (p. 6). The complex network of relationships within which teachers teach and students learn
Connectionist teachers (p. 97). Teachers who consistently make connections between different aspects of mathematics
Decile (p. 9). In New Zealand, a 1–10 system used by the Ministry of Education to indicate the socio-economic status of the communities from which schools draw their students; low-decile schools receive a higher level of government funding
Developmental progressions (p. 47). Sequential learning pathways categorised as a series of steps
Empirical evidence (p. 24). Data that has been collected systematically for research purposes
Equity (p. 9). The principle based on the belief that social injustices should be redressed by allocating resources according to need, not power; in education, this may mean, amongst other things, the provision of different pedagogical approaches depending upon the needs of the learners
Family Math (p. 171). A US initiative designed to develop parents’ skills so they can work with their children on their mathematics
Feed the Mind (p. 45). A media campaign funded by the New Zealand Ministry of Education and designed to support family involvement in children’s learning
High or low press for understanding (p. 121). Differing levels of cognitive engagement demanded of students by teachers for clarification of thinking
Kahao (p. 36). A festive necklace (Tongan)
Kōhanga reo (p. 9). Màori-medium early childhood centres
Kura kaupapa Màori (p. 10). Màori-medium schools (kura = school), based on a Màori philosophy of learning (see pp. 54–5)
Manipulatives (p. 133). Any concrete materials used by students to model mathematical relationships
Mathematical argumentation (p. 123). Presenting a case to support or refute a premise developed by mathematical thinking
Mathematical identity (p. 19). How a student sees him/herself as a learner and doer of mathematics
Metacognition (p. 38). The knowledge and processes involved in thinking about and regulating one’s own thinking, which is essential for reflecting, self-monitoring, and planning
Norms of participation (p. 54). The rules, spoken or unspoken, that govern the way students behave and contribute in the classroom
Number Framework (p. 109). A model, structured in 8 stages, showing how students typically develop understanding of number and number operations (New Zealand, NDP)
Number sense (p. 98). An understanding of the relationships, patterns, and fundamental reasonableness that lie behind all mathematical operations
Numeracy (p. 28). The ability to use mathematics effectively, fluently, and with understanding in a wide variety of contexts
Numeracy Development Project (NDP) (pp. 9, 17). The central part of the New Zealand Ministry of Education’s Numeracy Strategy, which has as its primary objective the raising of student achievement in numeracy through lifting teachers’ professional capability
NumPA (p. 9). A structured, diagnostic interview used by teachers to place students on the early stages of the Number Framework (New Zealand, NDP)
Open-ended tasks (p. 106). Tasks that require students to engage in problem definition and formulation before beginning to think about a solution
Pasifika students (p. 9). Students whose families have come from Sàmoa, Tonga, the Cook Islands, Niue, Tokelau, Tuvalu, and some other, smaller Pacific nations
Pedagogical Content Knowledge (p. 199). In this context, knowledge about mathematics and how to teach it as well as knowledge about how to understand students’ thinking about mathematics
Pedagogy (p. 5). The processes and actions by which teachers engage students in learning
Poi (p. 26). A small ball, often made of woven flax, on the end of a length of string; swung rhythmically by women when performing action songs (Màori)
QUASAR (p. 95). A programme developed to help urban students develop understanding of mathematical ideas through engagement with challenging mathematical tasks
Revoicing (p. 78). The repeating, rephrasing, or expansion of student talk in order to clarify or highlight content, extend reasoning, introduce new ideas, or move discussion in another direction
Scaffolding (p. 27). Temporary, structured support designed to move learners forward in their thinking
School–home or home–school partnership (p. 160). The deliberate nurturing of relationships between the school and the home, in the interests of better supporting student learning

Sociocultural practices (p. 19). Practices relating to the social and cultural aspects of participation in the classroom

Sociocultural theory (p. 24). The theory that learning arises out of social interaction

Socio-economic status (SES) (p. 30). Categorisation of individuals or communities, based on income, family background, and qualifications

Sociomathematical norms (pp. 61–62). Shared understandings of the processes by which students and teacher contribute to a mathematical discussion

Tasks (p. 94). Defined by Doyle (1983) as “products that students are expected to produce, the operations that students are expected to use to generate those products, and the resources available to students while they are generating the products”

Te ao Māori (p. 54). The Māori world

Te Poutama Tau (p. 59). The Numeracy Project (New Zealand) as developed for implementation in Māori-medium schools

Te Whāriki (p. 24). The New Zealand early childhood curriculum (for children aged 5 or under)

Tukutuku panels (p. 115). A Māori craft form consisting of ornamental lattice-work panels woven together with strips of flax into intricate designs

Waiata (p. 26). A song (Māori)

Whānau (p. 41). Extended family (Māori)

Wharekura (p. 9). Māori-medium secondary schools, which are based on a Māori philosophy of learning

Zone of Proximal Development (ZPD) (p. 36). Vygotsky (1986) describes the ZPD as the “distance between the actual development level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers”

**Abbreviations**

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<th>Abbreviation</th>
<th>Description</th>
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<tbody>
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<td>CGI</td>
<td>Cognitively Guided Instruction Project</td>
<td>pp. 17, 105</td>
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<tr>
<td>EAL</td>
<td>English as an Additional Language</td>
<td>p. 116</td>
</tr>
<tr>
<td>EFTPOS</td>
<td>Electronic Funds Transfer at Point of Sale</td>
<td>p. 115</td>
</tr>
<tr>
<td>EMI-4s</td>
<td>Enhancing the Mathematics of Four-Year-Olds</td>
<td>p. 28</td>
</tr>
<tr>
<td>ENRP</td>
<td>Early Numeracy Research Project</td>
<td>p. 158</td>
</tr>
<tr>
<td>EPPE</td>
<td>Effective Provision of Pre-school Education Project</td>
<td>p. 25</td>
</tr>
<tr>
<td>ERO</td>
<td>Education Review Office</td>
<td>p. 158</td>
</tr>
<tr>
<td>IAMP</td>
<td>Improving Attainment in Mathematics Project</td>
<td>pp. 18, 99</td>
</tr>
<tr>
<td>ICME</td>
<td>International Congress on Mathematics Education</td>
<td>p. 20</td>
</tr>
<tr>
<td>ICT</td>
<td>Information and Communication Technologies</td>
<td>p. 27</td>
</tr>
<tr>
<td>IEA</td>
<td>International Association for the Evaluation of Educational Achievement</td>
<td>p. 154</td>
</tr>
<tr>
<td>IMPACT</td>
<td>Increasing the Mathematical Power of All Children and Teachers</td>
<td>p. 73</td>
</tr>
<tr>
<td>MEP</td>
<td>Mathematics Enhancement Project</td>
<td>p. 60</td>
</tr>
<tr>
<td>NCEA</td>
<td>National Certificate of Educational Achievement</td>
<td>pp. 10, 66</td>
</tr>
<tr>
<td>NEMP</td>
<td>National Education Monitoring Project</td>
<td>p. 9</td>
</tr>
<tr>
<td>NNS</td>
<td>National Numeracy Strategy</td>
<td>p. 17</td>
</tr>
<tr>
<td>PISA</td>
<td>Program for International Student Assessment</td>
<td>p. 8</td>
</tr>
<tr>
<td>REPEY</td>
<td>Researching Effective Pedagogy in the Early Years</td>
<td>p. 25</td>
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<tr>
<td>RME</td>
<td>Realistic Mathematics Education</td>
<td>p. 113</td>
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<tr>
<td>TIMSS</td>
<td>Third International Mathematics and Science Study</td>
<td>p. 14</td>
</tr>
<tr>
<td>VAMP</td>
<td>Values and Mathematics Project</td>
<td>p. 58</td>
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