MINISTRY OF EDUCATION
Te Tāhuhu o te Mātauranga

Effective Pedagogy in Mathematics/Pāngarau

Best Evidence Synthesis Iteration [BES]

Glenda Anthony and Margaret Walshaw, Massey University

BES
ITERATIVE BEST EVIDENCE SYNTHESIS PROGRAMME
http://educationcounts.edcentre.govt.nz/goto/BES
New Zealand

This report is one of a series of best evidence synthesis iterations (BESs) commissioned by the Ministry of Education. The Iterative Best Evidence Synthesis Programme is seeking to support collaborative knowledge building and use across policy, research and practice in education. BES draws together bodies of research evidence to explain what works and why to improve education outcomes, and to make a bigger difference for the education of all our children and young people.

Each BES is part of an iterative process that anticipates future research and development informing educational practice. This BES follows on from other BESs focused on quality teaching for diverse learners in early childhood education and schools. Its use will be informed by other BESs, focused on teacher professional learning and development and educational leadership. These documents will progressively become available at: http://educationcounts.edcentre.govt.nz/goto/BES

Feedback is welcome at best.evidence@minedu.govt.nz

Note: the references printed in purple refer to a list of URLs in Appendix 2. These are a selection of potentially useful sources for teachers to engage more deeply with the range of issues raised in this best evidence synthesis iteration.

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About the writers

Glenda Anthony and Margaret Walshaw, both from the School of Curriculum and Pedagogy at Massey University, bring to this Best Evidence Synthesis (BES) decades of mathematics classroom teaching and educational research experience. They are acutely aware of the challenge that educators face in constructing a democratic mathematical community with which all students can identify. For them, making a positive difference to diverse learners' outcomes is a central educational issue. At the heart of their work is a concerted effort to illuminate how this issue is best addressed. In this synthesis, they report on the outcome of their deliberations over, and search for, what makes a difference for diverse learners in mathematics/pāngarau.

Advisory Group

A core Advisory Group membership was selected to provide expertise and critique in relation to the various focuses of the BES, including Māori and Pasifika learners, early childhood, primary and secondary sectors, and teacher education. The authors wish to thank the members of this group:

- Dr Ian Christensen (Massey University and He Kupenga Hao i te Reo)
- Dr Joanna Higgins (Victoria University of Wellington)
- Roberta Hunter (Massey University)
- Garry Nathan (Auckland University)
- Dr Sally Peters (Waikato University)
- Assoc. Prof. Jenny Young-Loveridge (Waikato University)

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Ministry of Education advisory team

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External quality assurance

Professor Paul Cobb from Vanderbilt University, US, has provided invaluable assistance. We would like to acknowledge his scholarly critique and thank him for his knowledgeable contribution to the synthesis.

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Forewords

International

Even the casual visitor is struck by the dramatic changes that have occurred in New Zealand in the last 15 years. I have tuned in to local media on each of my four visits to get an initial sense of people's current concerns and issues. Based on this narrow sampling, the New Zealand of 1991 was an immensely likeable country that had seen better days and was struggling to find its place in a rapidly changing world. Although innovation and experimentation appeared to be the watchwords of the day, there seemed to be an undercurrent of apprehension and anxiety as people attempted to cope with economic disruption. Today, New Zealand continues to be an immensely likeable place, but the visitor immediately notices a quiet, understated self-assurance. It has become a largely prosperous country that, in a very real sense, has reinvented itself as a leading information economy in an increasingly globalised world. Refreshingly for the visitor from the United States, there appears to be widespread belief that government will approach problems pragmatically and is capable of solving them. If the Iterative Best Evidence Synthesis Programme is representative of New Zealand government in action, this belief would appear to be well founded.

Put quite simply, the Iterative BES Programme is the most ambitious effort I have encountered that uses rigorous scientific evidence to guide the ongoing improvement of an education system at a national level. The programme has a strong pragmatic bent and is clearly grounded in the hard-won experience of synthesising research findings to inform both policy and teachers' instructional practices. Four aspects of the programme are particularly noteworthy. The first is the overriding commitment to make the development of the best evidence syntheses transparent. This commitment takes concrete form in the exacting evaluation and feedback process that all BES reports undergo at each phase of their development, from the initial identification of relevant bodies of research literature through to the final critique and revision of the report. This is in the best traditions of science, where claims are justified in terms of the means by which they have been produced.

The second notable characteristic is a mature view of evidence and an emphasis on methodological and theoretical pluralism. This is important, given that attempts have been made in a number of countries, including the United States, to legislate what counts as scientific research in education on the basis of ideological adherence to a particular methodology. In taking an inclusive approach, the Iterative BES Programme acknowledges that different types of knowledge are of greatest use to teachers and to policymakers. Teachers make pedagogical decisions on the basis of a detailed understanding of specific students in particular classrooms at particular points in time. Policymakers, in contrast, typically need knowledge of trends and patterns that hold up across classrooms to make decisions that affect large numbers of students and teachers in multiple schools. Different methodologies are appropriate for developing these equally important types of knowledge.

The third noteworthy characteristic of the Iterative BES Programme is its focus on the explanatory power and coherence of theories. Priority is given to theories that give insight into learning processes and the specific means of supporting their realisation in classrooms. This pragmatic criterion is important in a field where theoretical perspectives continue to proliferate.

The final notable characteristic of the programme is its explicit attention to the issues of language and culture. This emphasis is clearly critical if New Zealand teachers and policymakers are to address the inequities inherent in the disturbingly large gaps in school achievement between children of different ethnic and racial groups. In keeping with the tenet of methodological and theoretical pluralism, the Iterative BES Programme uses group categories such as socioeconomic status, ethnicity, and culture as key variables in *assessing* efforts to achieve

equity. However, it avoids stereotyping children of particular racial, ethnic, or language groups by acknowledging the complexity of individual identity when *explaining* inequities in children's learning opportunities. Furthermore, the programme emphasises ecological models of learning that link what is happening in classrooms both to the institutional contexts in which classrooms are located and to issues of race, culture, and language. It is here that the full ambition of the programme becomes apparent: few viable models of this type currently exist in education. The BES writers are therefore charged with the task of synthesising in the true sense of the term, that is, to combine disparate and sometimes fragmented bodies of research into a single, unified whole. At the risk of understatement, this is a formidable challenge.

The writers of this Best Evidence Synthesis of Effective Pedagogy in Mathematics/Pāngarau, Drs. Glenda Anthony and Margaret Walshaw, have risen to the challenge. They were charged with the daunting task of reviewing, organising, and synthesising all mathematics education research from the early childhood years through secondary school that relates classroom processes to student learning. On my reading, the resulting synthesis of over 600 research studies is directly relevant to teachers and will be educative for policymakers. The educative value of the report stems from Anthony and Walshaw's focus on what goes on in mathematics classrooms, thereby providing a window on the complexity of effective pedagogy. The forms of pedagogical practice that they identify as effective are ambitious because they involve high expectations for *all* children's mathematical learning. The goals at which these forms of pedagogy aim are best illustrated in chapter 7, A Fraction of the Answer, in which Anthony and Walshaw pull together the key insights of the proceeding chapters as they present an integrated series of cases that focus on the learning and teaching of fractions. As this chapter makes clear, the instructional goals for fractions are not limited to ensuring that children can add, subtract, multiply, and divide fractions successfully. Instead, the instructional objectives also focus on children's development of a deep understanding of fractions as amounts or quantities. At an elementary level, children who are coming to understand fractions as quantities know that $\frac{1}{6}$ is smaller than $\frac{1}{5}$ because there will be more pieces when something is divided into 6 pieces than into 5 pieces, so the pieces must be smaller. At a more advanced level, students will be able to describe real world situations that involve multiplying and dividing fractional quantities. More generally, ambitious pedagogy focuses on central mathematical ideas and principles that give meaning to computational methods and strategies.

Anthony and Walshaw's review of the relevant research indicates that central mathematical ideas and principles cannot be directly transmitted to children. However, the research also shows that discovery approaches that place children in rich environments and simply encourage them to inquire are also ineffective. Effective pedagogy is complex because it requires teachers to achieve a significant mathematical agenda by taking children's current knowledge and interests as the starting point. As Anthony and Walshaw clarify, these forms of pedagogy involve a distinctive orientation towards teaching. First and foremost, the emphasis is on building on students' existing proficiencies rather than filling gaps in students' knowledge and remediating weaknesses. As a consequence, the teacher's focus when planning for instruction is not on students' limitations but on their current mathematical competencies and interests, as these constitute resources on which the teacher can build. More generally, effective mathematical pedagogy places students' reasoning at the center of instructional decision making. As a consequence, the ongoing assessment of students' reasoning is an integral aspect of instruction, not a separate activity conducted after the fact to check whether goals for students' learning have been achieved. A key characteristic of accomplished teachers is that they continually adjust instruction, as informed by these ongoing assessments.

One of the strengths of Anthony and Walshaw's synthesis is that it provides the reader with a concrete image of what effective mathematical pedagogy looks like. Anthony and Walshaw emphasise that a respectful, non-threatening classroom atmosphere in which all students feel comfortable in making contributions is necessary but not, by itself, sufficient. As they document, the research findings indicate unequivocally that it is also essential that classroom activity

and discourse focus explicitly on central mathematical ideas and processes. The selection of instructional tasks is therefore critical. On the one hand, it is important that task contexts or scenarios are accessible to all students, regardless of cultural background. On the other, the teacher should be able to capitalise on students' solutions to support their development of increasingly sophisticated forms of mathematical reasoning. Thus, when designing and selecting tasks, the teacher has to take account both of students' current competencies and interests and their long-term learning goals. As Anthony and Walshaw discuss in chapter 5, an important way in which the teacher can build students' solutions is by introducing judiciously chosen tools and representations. A second, equally important way in which the teacher can capitalise on the potential of worthwhile mathematical tasks is to engage students in justification, abstraction, and generalisation (see chapter 4), by doing which they learn to speak the language of mathematics.

The image of effective mathematical pedagogy that emerges from Anthony and Walshaw's synthesis is of teaching as a coherent system rather than a set of discrete, interchangeable strategies. This pedagogical system encompasses:

- a non-threatening classroom atmosphere;
- instructional tasks;
- tools and representations;
- classroom discourse.

To see that these four aspects of effective pedagogy constitute a system, note that the way in which instructional tasks are realised in the classroom and experienced by students depends on the classroom atmosphere, the tools and representations available for them to use, and the nature and focus of classroom discourse. And because effective pedagogy is a system, it makes little sense to think of student learning as being *caused* by isolated teacher actions or strategies. It is for this reason that Anthony and Walshaw speak of mathematical learning being *occasioned* by teaching. In using this term, Anthony and Walshaw emphasise the teacher's proactive role in supporting students' development of increasingly sophisticated forms of mathematical reasoning.

In addition to highlighting the systemic character of effective mathematical pedagogy, Anthony and Walshaw make good on the charge to develop an ecological model of learning that links what is happening in the classroom to issues of race, culture, and language, and to the school contexts in which teachers develop and revise their instructional practices. A concern for issues of equity permeate the entire report but come to the fore in the discussion of school–home partnerships that take the diverse cultures of students and their families seriously and treat them as instructional resources.

Anthony and Walshaw make it clear that it is essential to view school contexts as settings for teachers' ongoing learning. In a very real sense, these settings mediate the extent to which high quality teacher professional development will result in significant changes in teachers' classroom practices. Anthony and Walshaw's synthesis documents that mathematics instruction that places students' reasoning at the center of instructional decision making is demanding, uncertain, and not reducible to predictable routines. The available evidence indicates that a strong network of colleagues constitutes a crucial means of support for teachers as they attempt to cope with these uncertainties and the loss of established routines. Consequently, there is every reason to expect that improvement in teachers' instructional practices and student learning will be greater in schools where mathematics teachers participate in learning communities whose activities focus on central mathematical ideas and how to relate them to student reasoning. The value of teacher learning communities in turn foregrounds the critical role of the principal as an instructional leader.

Historically, teaching and school leadership have been loosely coupled, with the classroom being treated as the preserve of the teacher while school leaders managed around instruction. Recent research findings demonstrate the limitations of this type of school organisation

in supporting the improvement of teaching on any scale. These findings also indicate that principals can play a key role in supporting the emergence of a shared vision of what effective mathematical pedagogy looks like and in supporting teacher collaboration that focuses on challenges central to the development of effective pedagogy. This alternative type of school organisation is characterised by reciprocal accountability. Teachers are accountable to principals for developing increasingly effective pedagogical practices and principals are accountable to teachers to create opportunities for their ongoing learning. Changes of this type in the relations between teachers and school administrators are far reaching and might be viewed as too radical. It is, however, sobering to note that previous large-scale efforts to improve the quality of classroom instruction have rarely produced lasting changes in teachers' practices. Research into educational leadership and policy indicates that this history is due in large part to the failure to take into account the institutional settings in which teachers develop and refine their instructional practices.

The broader policy and leadership literature strongly indicates that the improvement of mathematics instruction on the scale being attempted in New Zealand is not simply a matter of providing high quality teacher professional development. It also has to be framed as a problem for schools as educational organisations that structure the institutional settings in which teachers develop and revise their instructional practices. My reading of this Best Evidence Synthesis of Effective Pedagogy in Mathematics/Pāngarau is that Anthony and Walshaw have distilled valuable lessons from the available research, thereby positioning New Zealand educators to succeed where others have failed.

Paul Cobb

Professor of Mathematics Education
Vanderbilt University, Tennessee

Note: The second Hans Freudenthal Medal of the International Commission on Mathematical Instruction (ICMI) was awarded to Professor Paul Cobb in 2005, "whose work is a rare combination of theoretical developments, empirical research and practical applications. His work has had a major influence on the mathematics education community and beyond."

Early Childhood Education

This Best Evidence Synthesis of Effective Pedagogy in Mathematics/Pāngarau is a 'must read' for those in the early childhood sector who want an insight into what effective mathematical pedagogy looks like in an early childhood service. The synthesis acknowledges the vital role that quality early childhood education plays in the mathematical development of infants and young children. It also provokes early childhood teachers to reflect on practice: their mathematical awareness of the environment, the depth of their mathematical knowledge, and the importance of effective teaching and learning strategies that will support children's optimal engagement in mathematical experiences. The extensive, wide-ranging research information is effectively balanced by vignettes which involve the reader in meaningful mathematical experiences that illustrate the possibilities for supporting mathematical learning. Effective distribution of the synthesis would enhance teaching and learning outcomes in early childhood services.

Faith Martin
Director, Massey Child Care Centre

NZEI Te Riu Roa

NZEI Te Riu Roa welcomes this Best Evidence Synthesis of Effective Pedagogy in Mathematics/Pāngarau, particularly as it takes for its starting point the assertion that "all children can learn mathematics". This key message is at the heart of every teacher's commitment to the mathematical learning of his or her students.

The synthesis recognises the complexity of teaching, particularly given the diverse learning needs of the students in our classrooms and centres and the necessity for specialised knowledge of mathematics. But the writers consistently underline the power that teachers have to make a difference: "It is what teachers do, think and believe (that) significantly influences student outcomes."

A teacher's role, whether in a school or a centre, includes the design of activities that help students to construct meaning and think for themselves. To achieve such outcomes, teachers need to appreciate the part that mathematics plays in the world around them, what the big mathematical ideas are, and how the concepts that they teach fit in with those ideas. They need to know how to teach knowledge and skills, how to match new learning with students' prior knowledge, and which activities effectively encourage understanding and learning. Teachers also need to be conscious of developing attitudes and values. They need to create opportunities for their students to develop a critical eye and, in the context of this synthesis, a critical mathematical eye.

The primary purpose of the synthesis is to identify evidence that links pedagogical practice with effective mathematics outcomes for students. To achieve this, the writers have drawn on national and international research that contributes to our understanding of what works in mathematics education.

When reviewing the synthesis in its draft form, NZEI teachers were particularly pleased to read the chapter, Mathematics Practices Outside the Classroom, which they saw as contributing to a constructive environment and encouraging of good practice. The synthesis explores ways in which parents can contribute to their children's mathematical development and ways in which schools can strengthen links with the home. If teachers are to successfully fulfil expectations, such links are likely to be vital. Teachers were also pleased to see the importance of school leadership recognised.

NZEI sees the Effective Pedagogy in Mathematics/Pāngarau BES as being of great benefit to teachers, teacher educators, and policymakers. The research identified in the synthesis, together with the case studies and vignettes, has the potential to stimulate much constructive professional discussion. To maximise its potential for teachers, it will need to be accompanied by professional learning opportunities and time for reflection and discussion in the school or centre setting.

Irene Cooper
National President
Te Manukura
NZEI Te Riu Roa

Post Primary Teachers' Association

Tēnā koutou, tēnā koutou, tēnā tatou katoa.

PPTA welcomes this Best Evidence Synthesis of Effective Pedagogy in Mathematics/Pāngarau. It is the result of a very thorough process, inclusive of the expertise of practitioners. The final report reflects and caters to their realities, and provides some very interesting and thought-provoking reading for teachers themselves, and for those involved in the pre-service and in-service education of mathematics teachers. At the same time, the research highlights the shortage of outcomes-linked research evidence specific to secondary school mathematics teaching and we hope that as a result of this BES, New Zealand researchers will step up to fill this gap.

Debbie Te Whaiti
President
New Zealand Post Primary Teachers' Association

Teacher Educators

The Best Evidence Synthesis of Effective Pedagogy in Mathematics/Pāngarau succeeds in providing a systematic treatment of relevant outcomes-based evidence for what works for diverse learners in the New Zealand education system. One of the strengths of the document is the central positioning given by its authors to a broad notion of diversity.

Teacher educators, both initial and ongoing, will find that the BES is an invitation to engage—as teachers and as researchers—with a wide range of national and international studies. The document succeeds in preserving the complexity of pedagogical approaches through careful structuring and presentation. Well chosen classroom vignettes capture the essence of pedagogical issues for use in initial and ongoing teacher education. The CASEs are likely to prove particularly valuable for teachers by demonstrating how research can inform classroom practice.

The BES also presents a challenge to New Zealand researchers by identifying areas in which there is a paucity of outcomes-based evidence. Such evidence is scarce for Māori-medium mathematics classrooms. The senior secondary area is generally not well represented and a wider range of early childhood contexts needs to be investigated. The CASEs highlight for teacher educators the possibilities of writing up research projects undertaken as part of ongoing teacher education initiatives, and encourage them to gather further evidence to support practice.

The importance to mathematics education of the outcomes-based research evidence represented in this synthesis cannot be overstated. It is to be hoped that the value of the Iterative BES programme is widely recognised, and that it has the impact on policy and practice that it ought.

Joanna Higgins
Director, Mathematics Education Unit and Associate Director,
Jessie Hetherington Centre for Educational Research
Victoria University of Wellington

Māori-medium Mathematics

E nga mana, e nga reo, tēnā koutou katoa.

For the last 20 years, the teaching of pāngarau (mathematics) has played a significant role in the revitalisation of te reo Māori. The Effective Pedagogy in Mathematics/Pāngarau BES recognises the close relationship that exists between language and the learning and teaching of mathematics.

The BES identifies a range of major considerations and challenges for teachers and all those involved in Māori-medium education. The research makes it clear that mathematical outcomes for students are affected by a complex network of interrelated factors and environments, not just individual preferences or the language of instruction. By identifying the key elements in this network and discussing the relevant research, the writers have created what should prove a very useful resource.

The BES highlights the paucity of research into Māori-medium mathematics education, particularly in the area of teacher practice.

Tony Trinick
Māori-medium mathematics educator
Faculty of Education
The University of Auckland

Pasifika

E rima te'arapaki, te aro'a, te ko'uko'u te utuutu, 'iaku nei.
Under the protection of caring hands there's feeling of love and affection.

The Best Evidence Synthesis of Effective Pedagogy in Mathematics/Pāngarau has drawn together a comprehensive synthesis of evidence that relates to quality mathematics pedagogical practices. Its particular strength is that it provides stimulating and thought-provoking reading for a range of stakeholders and at the same time affirms that there is no one, specific, 'quality' pedagogical approach. Rather, it directs attention to many effective approaches which make a difference for all mathematics learners. The vignettes are an added strength; they make the theoretical structures they illustrate accessible to a wider audience.

The synthesis highlights the shortage of outcomes-linked research evidence concerning quality teaching and learning for Pasifika students at all levels of schooling. It also highlights the importance of a culture of care. How this translates into quality outcomes for Pasifika students requires the attention of New Zealand researchers.

Roberta Hunter
Senior Lecturer
School of Education Studies
Massey University, Albany Campus

Chief Education Adviser, BES

The Effective Pedagogy in Mathematics/Pāngarau BES sets out to uncover and explain the links between what we do in mathematics education and what the outcomes are for learners. The result is a valuable resource that can be used to enhance a wide range of outcomes for diverse learners. These include the ability to think creatively, critically, strategically and logically; mathematical knowledge; enjoyment of intellectual challenge; self-regulatory, collaborative and problem-solving skills; and the disposition to use, enjoy and build upon that knowledge throughout life.

The BES reflects the outstanding scholarly work and professional leadership of co-authors Drs Glenda Anthony and Margaret Walshaw of Massey University. They are the first to use the new *Guidelines for Generating a Best Evidence Synthesis* and follow the collaborative development process that is central to the Iterative BES Programme. They have consulted tirelessly and responsively with a wide range of early childhood teachers, primary and secondary teachers, principals, advisers, researchers, policy workers and teacher educator colleagues from across New Zealand, and with international colleagues. The Ministry of Education acknowledges and values all these contributions—and those of the formative quality assurers, whose affirmations and challenges have been so helpful in optimising the quality and potential usefulness of this BES.

The BES celebrates and returns to early childhood educators, teachers, teacher educators and researchers a record of their professional work, highlighting the complexity of that work, and suggesting how research evidence can be a valuable resource to inform their ongoing work and that of their colleagues. From the first vignette explaining how mathematical learning can be embedded in waiata (Māori song) and dance, the vignettes bring children's learning in mathematics to life. The underlying explanations and theoretical findings have the power to inform practice in ways that are relevant and responsive to the learners in any particular centre or classroom.

The challenge now is for us all is to use this resource in ways that will support further systemic development in mathematics education, with strengthened outcomes for diverse learners. In many cases, the BES will affirm what is already happening, but it will be the points of challenge that take us forward. Individual teachers have already engaged with the BES in its draft form, and some report remarkable insights and developments in their practice. But it is only through the wider and systemic development of the conditions that support effective practice for diverse learners that improvements will proliferate and become self-sustaining. The findings emerging from the outcomes-linked professional learning and development BESs[1,2] should be an invaluable resource in determining how to generate changed practice on such a scale.

Many teachers and early childhood educators have indicated that they want to read this BES for themselves, and to do this they need time. They need time to read, discuss and consider how they can use relevant BES findings in response to diagnostic information about the mathematical understandings of the children and young people they teach. They also need time to participate in professional learning communities. The Teacher Professional Learning and Development BES[3] finds that such participation doesn't guarantee better outcomes for students, but it is a consistent feature of teacher professional learning that does have a strong positive impact.

The same BES highlights the important role that external expertise with strong pedagogical content knowledge can play in facilitating and supporting changes in practice that impacts positively on student outcomes. Such expertise can be vital in engaging teachers' theories and challenging problematic discourses. The findings do, however, caution that 'experts' need more than good intentions—in the worst-case scenario, teacher professional development can actually impact negatively on student achievement. This finding calls for careful and iterative evaluation of the effectiveness of all professional development.

The teacher education community in New Zealand has already made a foundational contribution to this BES with its engagement in the research and development reported in this BES, and its advice to the BES writers. As the Teacher Professional Learning and Development BES[4] will show, some of our most effective professional development has been taking place as part of the Numeracy Development Projects (NDP)—with effect sizes twice those attained in England[5,6,7]. The primary and early childhood teachers' union, NZEI, confirms what the evaluation reports have been saying: that teachers who have been involved in the NDP value the transformational experiences this professional learning has afforded them. Two teachers from a Hawkes Bay school explained to me recently that, as a result of professional learning undertaken through the NDP, they have changed the way they work across the curriculum—they now listen more, are more diagnostic, and they place much more emphasis on children articulating and sharing their learning strategies. The dynamic, reflective, nation-wide learning community of researchers, teacher educators, teachers, and other educators created by the NDP and its Māori-medium counterpart, Te Poutama Tau, has been inspirational for BES.

If the mathematics BES is to serve New Zealand education well, the teacher education and research communities must make it a 'living' BES by building on the powerful insights and exemplars it makes available, addressing the gaps, and ensuring a cumulative and increasingly dynamic shared knowledge base about what works for learners in New Zealand education. To assist in this collaborative work, the New Zealand Council for Educational Research is creating a database of relevant New Zealand education theses. It has already built a database to support this document, with live links to the electronic version so that readers can quickly access either the full text or bibliographic details for some of the most helpful articles that have informed the synthesis. These links are also listed in the print version.

It is our hope that this BES will stimulate readers to let the Iterative Best Evidence Synthesis Programme know of other/new research and development that should feature in future iterations of the synthesis. Such research needs to clearly document demonstrated or triangulated links to student outcomes (see the Guidelines for Generating a Best Evidence Synthesis Iteration, found on the BES website[8]), and preferably show larger positive impacts on desired outcomes for diverse learners. We are especially seeking studies of research and development in New Zealand contexts, but we are also interested in information on overseas studies that show particularly large impacts on diverse learners. Please send details to best.evidence@minedu.govt.nz.

In the New Zealand context, where schools and centres are self managing, principals and centre leaders have a critical role to play in supporting their staff to realise the potential of this BES. The Teacher Professional Learning and Development BES indicates that, in the case of the most effective school-based interventions, principals and others in leadership roles have actively supported the development of a learning culture amongst their teachers.

For centuries, societies have required their education systems to sort children into successes and failures. Knowledge societies, such as our own, require much more. Our challenge is to ensure that all our children flourish as learners, strong in their own identities, and confident global citizens.

To achieve such goals, we need to value, build upon, and go beyond the craft practice traditions that require each teacher to 'rediscover the wheel'. The Effective Pedagogy in Mathematics/Pāngarau BES has been designed to serve as a resource and catalyst for strengthened practice, innovation, and systemic learning. By using it, and by making learner outcomes our touchstone, we can work together to give our children a mathematics education that prepares them well for the opportunities and challenges that will be their future.

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Authors' Preface

What is a Best Evidence Synthesis in Mathematics?

A best evidence synthesis draws together available evidence about what pedagogical approaches work to improve student outcomes in Mathematics/Pāngarau. This synthesis is part of the Iterative Best Evidence Synthesis (BES) Programme established late in 2003 by the Ministry of Education to deepen understanding of what works in education. The programme involves policy, research, and practice in collaborative knowledge building, aimed at maximising desirable outcomes for the diverse learners in the New Zealand education system.

This best evidence synthesis in Mathematics/Pāngarau plays a key role in knowledge building for New Zealand education. As a capability tool, it identifies, evaluates, analyses, and synthesises what the New Zealand evidence and international research tell us about quality mathematics teaching. It shows us how different contexts, systems, policies, resources, approaches, practices, and influences all impact on learners in different ways. Importantly, it illuminates what the evidence suggests can optimise outcomes for diverse mathematics learners.

The importance of dialogue

The development of this BES has been shaped by the *Guidelines for Generating a Best Evidence Synthesis Iteration* (Alton-Lee, 2004) and informed by dialogue amongst policy makers, educators, researchers and practitioners. Right from the very early stages of its development, the health-of-the-system perspective taken in this synthesis has ensured that we have listened to and responded to the viewpoints of a wide range of constituencies. Our interactions with these multiple communities have revealed to us the key roles that infrastructure, context, settings, and accountabilities play in a system that is functioning effectively for all its learners. Our various stakeholders have challenged us not only to produce better and more relevant educational research but to consider how this knowledge base might best be used. It is our hope that this discussion across sectors will be ongoing.

We have received a strong and positive response to the best evidence synthesis work from New Zealand's primary and post-primary teacher associations. Both have reported on how helpful the synthesis is to their core professional work. For example, the New Zealand Educational Institute (NZEI) writes: "In our view, the writers have drawn on national and international research which contributes to an understanding of what works in mathematics education; they have identified the significance of the context and ways in which to strengthen practice … We liked the … underpinning view that all children can learn mathematics" (p. 2). The representative for the Post Primary Teachers' Association at the Quality Assurance Day is reported as saying: "There are numerous wonderful ideas in the synthesis, and I found myself repeatedly jolted into possibilities for my own classroom resources." In addition, a group of initial and ongoing mathematics teacher educators have welcomed the "sophisticated treatment of diversity" and the way in which "the complexity of pedagogical approaches is preserved" (Victoria University of Wellington College of Education, 2006, p. 1).

Writing for multiple audiences

Our task was to make the findings of the synthesis accessible to and of benefit to a range of educational stakeholders. At one level of application, it is intended to provide a strengthened basis of knowledge about mathematics pedagogical practices in New Zealand today. The evidence it produces is expected to inform teacher educators within the discipline of mathematics education about effective pedagogical practice. At another level, the synthesis attempts to make transparent to policy makers and social planners an evidential basis for quality pedagogical approaches in mathematics. At a third level, the synthesis is expected to benefit practitioners and assist them in doing the best possible job for diverse learners in their classrooms.

Our approach to the "almost overwhelming task" (Cobb, 2006) of writing with several levels of application in mind has been to draw on both formal and informal approaches. We have offset the 'academic' language of the BES by including a series of vignettes that expand upon broad findings. We have received feedback from a range of sources that these vignettes bring the reality of classroom life to the fore and, in particular, do not minimise the complexities of actual practice. We hope that researchers, policy makers and practitioners alike will see in the vignettes theoretical tools that have been adapted and used by actual teachers.

The BES as a catalyst for change

This best evidence synthesis in mathematics does more than synthesise and explain evidence about what works for diverse learners. By bringing together rigorous and useful bodies of evidence about what works in mathematics, the project plays an important function as a catalyst for change. It is designed to help strengthen education policy and educational development in ways that effectively address both the needs of diverse learners and patterns of systemic underachievement in New Zealand education. It is written with the intent of stimulating activity across practitioners, policy makers, and researchers and so to strengthen system responsiveness to educational outcomes for all students.

The writers anticipate that reflection on the findings will lead to sustainable educational development that has a positive impact on learners. It will create new insights into what makes a difference for our children and young people. Reflection on the findings will also spark new questions and renewed, fruitful engagement with mathematics education. These new questions, in turn, will render the BES a snapshot in time—provisional and subject to future change.

Key features

Key features of the BES are:

- Its teacher orientation. Its view is towards a strengthened basis of knowledge about instructional practices that make a difference for diverse groups of learners.
- Its cross-sectoral approach. Its scope takes in the teaching of children in early childhood centres through to the teaching of learners in senior secondary school classrooms.
- Its inclusiveness. It documents research that reveals significant educational benefits for a wide range of diverse learners. It pays particular attention to the mathematical development of Māori and Pasifika students and documents research that captures the multiple identities held by New Zealand learners.
- Its breadth of search coverage. It reports on the characteristics of effective pedagogy, following searches through multiple national databases and inventories as well as masters' projects and theses. It provides comprehensive information about effective teaching as evidenced from small cases, large-scale explorations, and short-term and longitudinal investigations.
- Its local character. It makes explicit links between claims and bodies of evidence that have successfully translated the intentions and spirit of the Treaty of Waitangi. It identifies research relevant to the particular conditions and contexts in New Zealand, both in mathematics education in particular and in education in general, in relation to the principles and goals of *Te Whāriki* for early childhood settings and of *The Curriculum Framework*, for teachers in English or Māori-medium settings.
- Its global linkages. It connects local sources with the international literature. It identifies important Australian and international work in the area and evaluates that wide-ranging resource in relation to similarities and differences in cultures,

populations and demographics between the country of origin and New Zealand.

- Its responsiveness to concerns about democratic participation. It heeds the concern about the development of competencies that equip students for lifelong learning. This orientation coincides with the national mathematics curriculum objective of developing those knowledges, skills, and identities that will enable students to meet and respond creatively to real-life challenges.
- Its quality assurance measures. It is guided by principles of transparency, accessibility, relevance, trustworthiness, rigour, and comprehensiveness. These principles form the backdrop to the selection and systematic integration of evidence.
- Its strategic focus on policy and social planning. It uses a health-of-the-system approach to address one of the most pressing problems in education, provide a direction for future growth, and push effective teaching beyond current understandings.
- Its provisional nature. The project is an important knowledge-building tool, creating new insights from what has gone before, and will be updated in the light of findings from new studies. The findings are, above all, 'of the moment' and open to future change.

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Executive Summary

The *Effective Pedagogy in Mathematics/Pāngarau: Best Evidence Synthesis Iteration [BES]* was funded by a Ministry of Education contract awarded to Associate Professor Glenda Anthony and Dr Margaret Walshaw at Massey University. The synthesis is part of the Iterative Best Evidence Synthesis (BES) Programme, established by the Ministry of Education in New Zealand, to deepen understanding from the research literature of what is effective in education for diverse learners. The synthesis represents a systematic and credible evidence base about quality teaching in mathematics and explains the sort of pedagogical approaches that lead to improved engagement and desirable outcomes for learners from diverse social groups. It marks out the complexity of teaching and provides insight into the ways in which learners' mathematical identities and accomplishments are occasioned by effective pedagogical practices.

The search of the literature focused attention on different contexts, different communities, and multiple ways of thinking and working. Priority was given to New Zealand research into mathematics in early childhood centres and schools, both English- and Māori-medium. Personal networks enhanced the library search and facilitated access to academic journals, theses and reports, as well as other local scholarly work. The New Zealand literature was complemented by reputable work undertaken in other countries with similar population and demographic characteristics. Indices, both print and electronic, were sourced, and the search covered relevant publications within the general education literature as well as specialist educational areas. In the end, 660 pieces of research, ranging from very small, single-site studies to large scale, longitudinal, experimental studies, found their way into the report.

Key findings highlight practices that relate specifically to effective mathematics teaching and to positive learning and social outcomes in centres/kōhanga and schools/kura. The findings stress the importance of interrelationships and the development of community in the classroom. They also reveal that both the cognitive and material decisions made by teachers concerning the mathematics tasks and activities they use, significantly influence learning. The findings demonstrate the importance of children's early mathematical experiences and stress that constituting and developing children's mathematical identities is a joint enterprise of teacher, centre/school, and family/whānau.

Key findings

In this section, key findings are organised and presented according to five themes: the key principles underpinning effective mathematics teaching, the early years, the classroom community, the pedagogical task and activity, and educational leadership and centre–home and school–home links.

Key principles underpinning effective mathematics teaching

Teachers who enhance positive social and academic outcomes for their diverse students are committed to teaching that takes students' mathematical thinking seriously. Their commitment to students' thinking is underpinned by the following principles:

- an acknowledgement that all students, irrespective of age, have the capacity to become powerful mathematical learners;
- a commitment to maximise access to mathematics;
- empowerment of all to develop mathematical identities and knowledge;
- holistic development for productive citizenship through mathematics;
- relationships and the connectedness of both people and ideas;
- interpersonal respect and sensitivity;
- fairness and consistency.

The early years

Young children are powerful mathematics learners. Quality teaching guarantees the development of appropriate relationships and support as well as an awareness of children's mathematical understanding. Research has consistently demonstrated how a wide range of children's everyday activities, play and interests can be used to engage, challenge and extend children's mathematical knowledge and skills. Researchers have found that effective teachers provide opportunities for children to explore mathematics through a range of imaginative and real-world learning contexts. Contexts that are rich in perceptual and social experiences support the development of problem-solving and creative-thinking skills.

There is now strong evidence that the most effective settings for young learners provide a balance between opportunities for children to benefit from teacher-initiated group work and freely chosen, yet potentially instructive, play activities. Opportunities for learning mathematics typically arise out of children's everyday activities: counting, playing with mathematical shapes, telling time, estimating distance, sharing, cooking, and playing games. Teachers in early childhood settings need a sound understanding of mathematics to effectively capture the learning opportunities within the child's environment and make available a range of appropriate resources and purposeful and challenging activities. Using this knowledge, effective teachers provide scaffolding that extends the child's mathematical thinking while simultaneously valuing the child's contribution.

The classroom community

Research has shown that opportunities to learn depend significantly on the community that is developed within centres and classrooms. Thus, people, relationships, and classroom environments are critically important. Whilst all teachers care about student engagement, research quite clearly demonstrates that pedagogy that is focused solely on the development of a trusting climate does not get to the heart of what mathematics teaching truly entails. Teachers who truly care about their students have high yet realistic expectations about enhancing students' capacity to think, reason, communicate and reflect upon their own and others' understandings. Research has provided conclusive evidence that effective teachers work at developing inclusive partnerships, ensure that the ideas put forward by learners are received with respect and, in time, become commensurate with mathematical convention and curricular goals.

Studies have provided conclusive evidence that teaching that is effective is able to bridge students' intuitive understandings and the mathematical understandings sanctioned by the world at large. Language plays a central role. Mathematical language involves more than vocabulary and technical usage; it encompasses the ways that expert and novice mathematicians use language to explain and to justify concepts. The teacher who has the interests of learners at heart ensures that the home language of students in multilingual classroom environments connects with the underlying meaning of mathematical concepts and technical terms. Teachers who make a difference are focused on shaping the development of novice mathematicians who speak the precise and generalisable language of mathematics.

Mathematics teaching for diverse learners creates a space for the individual and the collective. Whilst many researchers have shown that small-group work can provide the context for social and cognitive engagement, others have cautioned that students need opportunities and time to think and work quietly away from the demands of a group. There is evidence that some students, more than others, appear to thrive in class discussion groups. Many students, including limited-English-speaking students, are reluctant to share their thinking in class discussions. Research has also shown that an over-reliance on grouping according to attainment is not necessarily productive for all students. Teachers who teach lower streamed classes tend to follow a protracted curriculum and offer less varied teaching strategies. This pedagogical

practice may have a detrimental effect on the development of a mathematical disposition and on students' sense of their own mathematical identity.

Pedagogical tasks and activities

From the research, it is evident that the opportunity to learn is influenced by what is made available to learners. For all students, the 'what' that they do is integral to their learning. The 'what' is the result of sustained integration of planned and spontaneous learning opportunities made available by the teacher. The activities that teachers plan, and the sorts of mathematical inquiries that take place around those activities, are crucially important to learning. Effective teachers plan their activities with many factors in mind, including the individual student's knowledge and experiences, and the participation norms established within the classroom. Extensive research in this area has found that effective teachers develop their planning to allow students to develop habits of mind whereby they can engage with mathematics productively and make use of appropriate tools to support their understanding.

Choice of task, tools, and activity significantly influences the development of mathematical thinking. Quality teaching at all levels ensures that mathematical tasks are not simply 'time fillers' and is focused instead on the solution of genuine mathematical problems. The most productive tasks and activities are those that allow students to access important mathematical concepts and relationships, to investigate mathematical structure, and to use techniques and notations appropriately. Research provides sound evidence that when teachers employ tasks for these purposes over sustained periods of time, they provide students with opportunities for success, they present an appropriate level of challenge, they increase students' sense of control, and they enhance students' mathematical dispositions.

Effective teaching for diverse students demands teacher knowledge. Studies exploring the impact of content and pedagogical knowledge have shown that what teachers do in classrooms is very much dependent on what they know and believe about mathematics and on what they understand about the teaching and learning of mathematics. Successful teaching of mathematics requires a teacher to have both the *intention* and the *effect* to assist pupils to make sense of mathematical topics. A teacher with the intention of developing student understanding will not necessarily produce the desired effect. Unless teachers make good sense of the mathematical ideas, they will not have the confidence to press for student understanding nor will they have the flexibility they need for spotting the entry points that will move students towards more sophisticated and mathematically grounded understandings. There is now a wealth of evidence available that shows how teachers' knowledge can be developed with the support and encouragement of a professional community of learners.

Educational leadership and links between centre and home/school

Facilitating harmonious interactions between school, family, and community contributes to the enhancing of students' aspirations, attitudes, and achievement. Research that explores practices beyond the classroom provides insight into the part that school-wide, institutional and home processes play in developing mathematical identities and capabilities. There is conclusive evidence that quality teaching is a joint enterprise involving mutual relationships and system-level processes that are shared by school personnel. Research has provided clear evidence that effective pedagogy is founded on the material, systems, human and emotional support, and resourcing provided by school leaders as well as the collaborative efforts of teachers to make a difference for all learners.

Teachers who build whānau relationships and home–community and school–centre partnerships go out of their way to facilitate harmonious interactions between the sectors. There is convincing evidence to suggest that these relationships influence students' mathematical development. The home and community environments offer a rich source of mathematical experiences on which to build centre/school learning. Teachers who collaborate with parents, families/whānau and

community members come to understand their students better. Parents benefit too: through their purposeful involvement in school/centre activities, by assisting with homework, and in providing suitable games, music and books, they develop a greater understanding of the centre's or school's programme. Their involvement also provides an opportunity to scaffold the learning that takes place within the centre or school.

Overall key findings

This Best Evidence Synthesis examines the links between pedagogical practice and student outcomes. Consistent with recent theories of teaching and learning, it finds that quality teaching is not simply a matter of 'knowing your subject' or 'being born a teacher'.

Sound subject matter knowledge and pedagogical content knowledge are prerequisites for accessing students' conceptual understandings and for deciding where those understandings might be heading. They are also critical for accessing and adapting task, activities and resources to bring the mathematics to the fore.

The importance of building home–community and school–centre partnerships has been highlighted in a number of studies of effective teaching.

Early childhood centre researchers have provided evidence that the most effective settings offer a balance between opportunities for children to benefit from teacher-initiated group work and freely chosen, yet potentially instructive, play activities.

Within centres and classrooms, effective teachers care about their students and work at developing interrelationships that create spaces for learners to develop their mathematical and cultural identities.

Extensive research on task and activity has found that effective teachers make decisions on lesson content that provide learners with opportunities to develop their mathematical identities and their mathematical understandings.

Studies have provided conclusive evidence that teaching that is effective is able to bridge learners' intuitive understandings and the mathematical understandings sanctioned by the world at large.

Gaps in the literature and directions for future research

The synthesis provides research information about effective mathematics teaching. Although the scope of researchers' studies is broad and far-reaching, a number of gaps in the literature are apparent. Research has so far provided only limited information about effective teaching in New Zealand at the secondary school level. Additionally, there is little reported research that focuses on quality teaching for Pasifika students. Few researchers in New Zealand are exploring mathematics in early childhood centres. The New Zealand literature lacks longitudinal, large-scale studies of teaching and learning. Also missing are studies undertaken in collaboration with overseas researchers. Such research is crucially important for understanding teachers' work and the impact of curricular change. The scholarly exchange of ideas made possible through joint projects with the international research community contributes in numerous ways to the capability of our local researchers.

It is important to keep in mind that, as a knowledge building tool, the synthesis provides insights based on what has gone before. A snapshot in time, it is subject to change as new kinds of evidence about quality teaching become available. Important mathematics initiatives are underway in New Zealand schools and centres. The Numeracy Development Projects, new assessment methods, projects involving information technology, and a greater focus on statistics in the curriculum are just three examples of changes that are currently taking place. All new initiatives require research that monitors and evaluates their introduction and 'take up' by centres/schools and the changes in teaching and learning that take place as a result. Such research is necessary to guide future directions in schools, educational policy, and curriculum design.

5. Mathematical Tasks, Activities and Tools

Introduction

Within classrooms involving mathematical communities of practice, teachers need support to achieve their mathematical agenda. This chapter discusses how this support manifests itself in the form of instructional tasks and the tools available for solving those tasks.

> Research in New Zealand is increasingly showing that task design plays a central role in structuring and developing an effective learning community. The social and the cognitive are not distinct domains in practice, but are integrated and embedded in task and activity design and classroom organisation. (Alton-Lee, 2003, p. 27)

In the mathematics classroom, it is through tasks, more than any other way, that opportunities to learn are made available to students. Tasks are defined by Doyle (1983) as the "products that students are expected to produce, the operations that students are expected to use to generate those products, and the resources available to students while they are generating the products" (p. 161). Tasks and their associated activity are hugely significant in determining how students come to understand mathematics (Doyle, 1988). The mathematical tasks with which learners engage determine not only what substance they learn, but how they come to think about, develop, use, and make sense of mathematics:

> the cumulative effect of students' experience with instructional tasks is students' implicit development of ideas about the nature of mathematics—about whether mathematics is something they personally can make sense of, and how long and how hard they should have to work to do so. (Stein, Smith, Henningsen, & Silver, 2000, p. 11)

In considering tasks, we look first at how the cognitive activity associated with mathematical thinking can be supported by task purpose and design. Second, we consider the relationship of the task to the learner, providing evidence of the importance of teacher knowledge and expectations in 'hearing' the competencies of their students and building from them. We also consider how tasks are mediated by pedagogical affordances and constraints and the participation norms of the classroom. Third, we provide evidence of how teachers can support the use of tools as resources or learning supports. We see that it is not the tool (or the inscription) in isolation that offers support for the teacher; rather, it is the learners' use of the tool and the meanings that develop as a result of this activity. In this way, the tool is not seen as standing apart from the activity of the learner. As in previous chapters, we acknowledge the critical role of teacher knowledge in the complex instructional dynamic.

Tasks that are problematic and have a mathematical focus provide opportunities for mathematical thinking

Mediated by communities of practice and related mathematical and social norms, tasks introduce important mathematical ideas and provide opportunities for learners to engage in a range of thinking practices (Marton & Runesson, 2004). At every level, this requires "the development of knowledge of concepts, techniques, notations, and relationships; recognising them in familiar and unfamiliar forms; recalling facts, names, procedures; using procedures fluently and accurately; the ability to shift between methods and representations; applying knowledge to solve problems, possibly transforming it to do so; and creating generalisation, abstractions, images and methods" (Watson, 2004, p. 364).

Given the range of cognitive and metacognitive demands within the various strands of mathematics—for example, mastering early computational procedures, generalising number structure, visualising three-dimensional shapes, interpreting statistical data, imagining limits

in calculus, and dealing with complex numbers—tasks vary in format and purpose. Tasks that require students to engage in complex and non-algorithmic thinking promote exploration of connections across mathematical concepts (Stein, Grover, & Henningsen, 1996); tasks that require students to model their thinking promote reflection (Fraivillig, Murphy, & Fuson, 1999); tasks that require students to discern invariants and variation, and structure, promote generalisation (Watson & Mason, 2005); tasks that require students to interpret and critique data promote the disposition of 'scepticism' (Chatfield, 1998); tasks that require students to 'notice and wonder' promote the disposition of curiosity (Shaughnessy, 1997); and tasks that provide opportunities for 'mathematical play' promote conjecture and exploration (Holton, Ahmed, Williams, & Hill, 2001). However, according to Hiebert et al. (1996), tasks should share some commonality: they should be problematic for the learner and leave a mathematical 'learning residue' (Davis, 1992). Most importantly, this residue should consist of (a) insight into the structure of mathematics and (b) strategies or methods for solving problems.

Students who engage in meaningful mathematical tasks are potentially able to treat situations as problematic: something they need to think about, not simply a disguised way of practising already-demonstrated algorithms. To engage in problem-based tasks, students must impose meaning and structure, make decisions about what to do and how to do it, and interpret the reasonableness of their actions and solutions (Holton, Spicer, Thomas, & Young, 1996). Utilising tasks that have more than one solution strategy and which can be represented in multiple ways requires that students communicate and justify their procedures and understanding (Carpenter, Franke, & Levi, 2003). Tasks that fit these criteria "leave behind something of mathematical value" to the learner (Hiebert et al., 1997).

The New Zealand Numeracy Development Project (NDP) advocates problem-based tasks that focus on students' sense-making activities. Evaluation reports suggest that tasks that require students to justify their solution strategies and reflect on their thinking support student gains in computational proficiency. Teachers have given such tasks credit for creating more positive learning environments (Higgins, Bonne, & Fraser, 2004; Thomas & Tagg, 2004; Thomas & Ward, 2002). From an action research study involving two year 9 classes, Holton, Anderson, Thomas, and Fletcher (1999) report considerable improvement in the performance of a class of traditionally low-attaining students as a result of increased opportunities to engage in problem-solving activities.

Large-scale empirical studies of educational change in the US also link significant achievement gains to changes in classroom practices centred on enquiry-based problem-solving approaches (e.g., Swanson & Stevenson, 2002; Thomas & Senk, 2001). Balfanz, MacIver, and Byrnes (2006) researched the implementation of the Talent Development (TD) Middle School mathematics programme in high-poverty schools. The programme combined evidence-based mathematics reforms with a focus on problem-solving skills as opposed to routine mathematical procedures. Tracking the first four years, the study found that across all levels of the achievement spectrum, students from the TD classes outperformed control schools on multiple measures of achievement. The average effect size by the end of middle school was .24.[1]

One of the significant features of these large-scale reform programmes has been teacher professional development with a focus on curriculum development and planning. Stein et al.'s (1996) analysis of a sample of 144 tasks within the QUASAR middle-school programme found that nearly three-quarters of the tasks required students to engage in high-level cognitive processes—either the active "doing of mathematics" (40%) or the use of procedures with connections to concepts, meaning, or understanding (34%). Eighteen percent of the tasks focused on the use of procedures without making connections to concepts, meaning, or understanding. These tasks possibly fulfilled the intention of practice and consolidation—a necessary feature of mathematics learning (Anthony & Knight, 1999b). Overall, the researchers concluded that the tasks embodied many of the characteristics that support students' capacity to think and reason about mathematics in complex ways. This conclusion was strengthened by Stein's (2001) review of a range of studies that provided evidence that the move to a problem-

solving approach in mathematics teaching has been associated with "increases in student performance on assessment tasks that measure students' capacity to think, reason and communicate" (p. 112).

While research has shown that quality teaching focuses on the mathematical aspect of the task (Blanton & Kaput, 2005), teachers' attempts to make mathematics interesting are sometimes at the expense of accuracy and meaning (Christensen, 2004). Rubick's (2000) research into a statistical investigation carried out by a group of New Zealand year 7 and 8 students notes that the students were able to select an investigation (e.g., eye colour) that focused on counting data sets rather than the intended exploration of relationships within data. The teachers of middle grades in Moyer's (2002) study reported that they often used manipulatives to 'have fun', an activity which they distinguished from their 'real maths' instruction. The tension between the goal of creating a positive climate in the classroom and the need to foster mathematical understanding is illustrated in the following profile of Linda Arieto, a Puerto Rican teacher who shares a cultural and linguistic background with her students.

Linda's Lesson

Linda's lesson [on graphing] challenged traditional approaches to mathematics instruction in a number of ways. First, her lesson involved statistics ... Second, the nature of the graphing task centered on a real-world application of mathematics: collecting, organizing, and describing actual data. Finally, this lesson actively involved students in working collaboratively. Students worked in pairs and were thoroughly engaged in the task ... When asked about the purposes of the matchbook graphing lesson, Linda explained she wanted to "just expose them to graphing." Her ongoing goal was for students to "develop a tremendous love for maths," a love she was unable to cultivate for herself. Perhaps for this reason Arieto steered away from emphasising accuracy and validity in the graphing work ... Her emphasis on "having fun" was in keeping with her goal that students enjoy school and therefore continue to attend in a city region where dropout rates can be as high as 85% by the high school years. However, Linda seemed to miss the potential to capture students' interest by using mathematical tasks that fostered important conceptual learning, maintained high standards, and were challenging and engaging.

From Cahnmann and Remillard (2002)

Other studies report similar tensions. Bills and Husbands (2005) report on the practice of a secondary school mathematics teacher's efforts to consciously negotiate between her espoused values within mathematics education and education more generally. A tension existed between her desire to develop her students' sense of social well-being alongside their sense of mathematical well-being. As she perceived it, her fundamental role was to develop students' confidence and give them a sense of what they were capable of achieving. However, a pedagogy focused principally on the development of a non-threatening learning climate does not get to the heart of what mathematics teaching really entails (Connor & Michaels, 1996).

The organisation of social interaction within group settings can also override the mathematical focus of a task. Higgins (1998) found that group settings provided by New Zealand junior primary school teachers frequently did not involve carefully structured tasks, nor did they have classroom norms that supported student engagement with mathematical ideas or concepts. The following episode from Higgins' research illustrates the tension between a teacher's need to manage her class and her desire to promote learning opportunities within the context of a group task.

The Jigsaw Puzzle

The teacher at Sapphire School describes the use of independent tasks and group work in terms of an organisational device: *one where you don't need to sit down and teach them too much ... because*

it's pretty self-explanatory. Because often you don't get time to sit down and teach the independent activities separately from the others.

In describing an episode of an independent group activity involving jigsaw completion, Higgins notes that two six-year-old girls, Carol and Suzanne, started out by randomly picking up the loose pieces and trying to fit them together before placing them on the board. The muted colour of the pieces provide little clue as to the position of the pieces. After a short while Suzanne remarked, "Maybe a bit hard for us". The girls then tried to work from a corner piece but after 5 minutes on the puzzle, with no solution in sight, the girls played with the pieces, laughing and singing as they placed the pieces in the wrong place. After an interval of off-task behaviour, the girls unsuccessfully attempted to solicit some peer help. Another peer, Catherine, was approached. While they did get much of the puzzle done with Catherine's help, this was a less than positive experience. Catherine derided Carol and Suzanne's efforts: "*It's the other way derbrain*," and proceed to take over much of the completion of the puzzle.

Higgins' conclusion that Suzanne and Carol were unlikely to have learned any new strategies for jigsaw puzzle solving through this experience was in direct contrast to the perceptions of the teacher. Her reflection of the episode, during which she was teaching another group was: [*They*] *sat down and did it and they really discussed it and they really tried to solve it and they—they worked really hard on it and they finished it and it took them a LONG time but they worked really well together. … it's good to see that sort of thing happening where the children … do a mistake and they actually have to work out what's gone wrong … it's good to see them make mistakes and then work it out.*

Higgins suggests that the over-reliance on working with one small teacher-led group can potentially result in lost opportunities for mathematical learning. The jigsaw puzzle activity was intended to support the students' knowledge and skills about mathematical relations. However, apart from the mention of the concept of "fit," the puzzle appeared to be interpreted by the girls, and by a peer, as a measure of competence. In order to focus the students on the mathematical nature of the activities Higgins argues for a greater use of whole-class introductions, including discussion of possible strategies, negotiated ways in which children can assist each other, and plenary sessions which involve student and teacher reflection.

From Higgins (1997)

Likewise, Stein (2001) reports instances of displaced learning in co-operative tasks that have been insufficiently structured to engage students with mathematical ideas. Without an explicit focus on the mathematical structure and processes, mathematics learning may be incidental. Instead of marginalising tasks as mathematical 'field trips' or enrichment activities which occur in isolation, Blanton and Kaput (2005) argue that teachers must expect mathematical thinking to be "woven into the daily fabric of instruction" (p. 440).

The importance of attending to the structure of mathematics is emphasised in the influential study, *Effective Teachers of Numeracy* (Askew, Brown, Rhodes, Johnson, & Wiliam, 1997). From this study, Askew and colleagues conclude that highly effective teachers—referred to as *connectionist* teachers—consistently attend to connections between different aspects of mathematics, for example addition *and* subtraction or fractions, decimals, *and* percentages.

> I think you've got to know that they are inverse operations. Those two (addition and subtraction), and those two (multiplication and division) are linked, because when you are solving problems mentally you are all the time making links between multiplication, division, addition and subtraction. (Barbara, in Askew, 1999, p. 99)

Mulligan, Mitchelmore, and Prescott's (2005) research examined 103 first grade students' representations as they solved 30 tasks across a range of mathematical content domains such as counting, partitioning, patterning, measurement, and space.

Patterns

Mulligan and colleagues found that early school mathematics achievement was strongly linked with the child's development and perception of mathematical structure. Low achievers focused on superficial characteristics of problems; they consistently showed lack of attention to the mathematical or spatial structure. In contrast, high achievers were able to draw out and extend structural features, and demonstrated strong relational understanding in their responses. Figures 5.1 and 5.2 show two responses to a triangular pattern task in which the pattern was reconstructed from memory and extended, and illustrate clear differences in the development of the two children concerned over an 18-month period. In comparison to the high-achieving child's advances in structural representations, the low-achieving child demonstrates only partial awareness of patterns, and no developmental growth in structure.

Fig. 5.1. Structural stage development of high-achieving child

Fig 5.2. Structural stage development of low-achieving child

The researchers argue that low achievers may benefit from access to tasks that assist them in visual memory and recognition of basic mathematical and spatial structure in objects, representations and contexts. Assisting children to visualise and record simple spatial patterns accurately could potentially lead to much broader improvements in children's mathematical understanding.[2]

From Mulligan, Mitchelmore, and Prescott (2005)

Research studies have found that children's exploration of mathematical structure within early numeracy provides an effective bridge from numerical to early algebraic thinking. Carpenter, Franke, and Levi's (2003) research presents classroom episodes in which young children productively explore the interface between arithmetic and algebra. The focus of their research is the use of conjecture, or claims that seem plausible but which are not yet established (e.g., when you add an odd number to another odd number, the answer is an even number). In the New Zealand context, Irwin and Britt (2005) found that students in the NDP solved numerical problems that required manipulation more successfully than did students who had not participated in the project. The development of a "flexible array of skills for manipulating arithmetical relations in ways that exhibit number sense as well as operational sense where students develop understanding of flexible numerical structures involving the four arithmetic operations" (Irwin & Britt, p. 182) appears to facilitate the acquisition of algebraic thinking. While this study reported a significant socio-economic status (SES) effect, another study (Irwin & Britt, 2004), involving two low-decile schools and one middle-decile school, reported no SES effect on student achievement.

In a year-long investigation into the characteristics of instructional practice that support the development of algebraic reasoning, a third grade teacher's practices were analysed. The following vignette illustrates how the teacher, June, instead of implementing pre-designed algebraic tasks as stand-alone activities, incorporated opportunities for algebraic reasoning[3] into regularly planned—and what may have been more arithmetically focused—instruction.

Algebraic Treatment of Number

In the following episode June treated numbers in an algebraic way, that is, as a placeholder that required students to attend to structure rather than rely on the computation of specific numbers. In the interchange, June challenges a student's use of an arithmetic strategy to deduce that 5 + 7 was even:

June:	How did you get that?
Tony:	I added 5 and 7 and then I looked over there [pointing to a visible list of even and odd numbers on the wall] and saw that it was even.
June:	What about 45678 + 85631? Odd or even?
Jenna:	Odd.
June:	Why?
Jenna:	Because 8 and 1 is even and odd, and even and odd is odd.

The introduction of large numbers required the students to think in terms of even and odd properties to determine parity. In doing so, the researchers maintain that June used numbers as placeholders, or variables, for any odd or even numbers. Moreover, the researchers noted that in using numbers algebraically the teacher was able to avoid the semiotic complications of using literals (e.g., 2n + 1 for some integer n) to represent arbitrary even and odd numbers. This illustrates how the abstractness of numbers gets built as students work with particular quantities, and how the teacher can set the stage for the next move—the formal expression of the generalisation.

From Blanton and Kaput (2005)

In the secondary school context, researchers from the Improving Attainment in Mathematics Project (IAMP) attributed improvements in students' mathematical attainment to teachers and learners focusing on the development of ways to think with, and about, key ideas in mathematics. Like researchers in the primary context, De Geest, Watson, and Prestage (2003) note the importance of connections within mathematics:

> The 'contents of the subject-matter domain' are deeply connected within themselves through mathematical structure, and that enculturation into mathematical thinking involves becoming fluent with constructing, creating and navigating similar or isomorphic structures, that is, being intimately attuned to the ways in which mathematics is internally connected. (p. 306)

Watson's (2002) experiences with low attainers convinced her that "some potentially powerful mathematical talents of these students were unrecognised and unused in the teaching of mathematics" (p. 472). In advocating the benefits of pedagogy focused on awareness of mathematical structure, Watson provides the following example to illustrate these students' engagement in mathematical thinking.

Mathematical Talent

Secondary students from a low-attaining class grouping had been using flow diagrams to calculate the outputs of compound functions such as shown in figure 5.3.

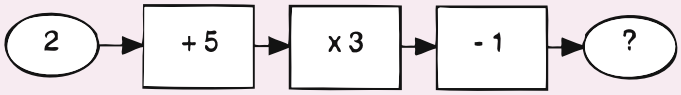


Fig. 5.3. A flow diagram

In response to a request to make up some hard examples of their own, most students provided examples with more operations and bigger numbers. However, one student suggested constructing problems in which the operations and output are known and the input has to found. Another student provided an example in which input and output were given but the last operation was missing. According to Watson, these two students were working with the relations rather than the numbers and operations. They saw the structure of the problem as something they could vary, rather than following the template of teacher-given questions. The generation of the special examples enabled the students to shift to more complex levels of mathematical thinking.

From Watson (2002)

Watson (2002) readily acknowledges the difficulties of working with low-attaining students. However, as a result of her research findings, she offers three principles for working with low attainers, based on student task construction and interactive strategies that are focused on mathematical thinking:

1. Their attention can be drawn to structure through observing patterns which go across the grain of work.
2. They can be asked to exemplify, and hence get a sense of structure, generality, and extent of possibilities.
3. They can be prompted to articulate similarities in their work, and hence be prompted to represent similarities in symbols.

Confirming the importance of mathematical focus, Watson (2003) claims that teachers can learn much about students' responses to tasks by thinking about what the task affords in terms of activity, what is constrained, and what attunements or patterns of participation are brought to bear in the activity that is generated by the task. Watson, along with other researchers (e.g., Marton & Tsui, 2004; Runesson, 2005), suggests that teachers can focus and refine the opportunities for learning mathematics by controlling the amount of variation permitted in any task or series of tasks. While each mathematical task affords opportunities to learn, it is by limiting variation to the feature on which it is hoped students will focus and by inviting conjecture and generalisation that students can be directed to the construction of mathematical meaning.

Marzano, Pickering, and Pollock (2001) analysed numerous classroom-based research studies. They suggest that explicit guidance in identification of similarities and differences, accompanied with opportunities for students to independently identify similarities and differences, "enhances students' understanding of and ability to use knowledge" (p. 15). To illustrate the power of task variation, Runesson (2005) provides the following vignette (cited in Jaworski, 1994) that involves an exchange between two girls working on a task relating to volume and surface area.

Claire's Task Variation

In the episode, the teacher, Claire, has set up a provocative situation by challenging her students with an apparent contradiction.

Cl: We're saying, volume, surface area and shape, three sorts of variables, *variables.* And you're saying, you've fixed the shape—it's a *cuboid.* And I am going to say to you, hm.

Cl: I'll be back in a minute.

Cl: *That's* a cuboid. [She picks up a tea packet]. *That* is a cuboid. [She picks up an electric bulb packet.] [She picks up a meter ruler.] *This* is a cuboid. [She looks around their faces. Some are grinning.] And you are telling me that those are all the same shape? [Everyone grins.]

R: Well, no-o. They've all got six separate sides though.

Cl: They've all got six sides. But I wouldn't say that that is the same shape as that. [She compared the meter rule with the bulb box.]

R: No-o.

Cl: Why not?

D: Yes you would ... [There is an inaudible exchange between the girls D and R.]

Cl: What is different? [Hard to hear responses include the words *size* and *longer.*]

Cl: Different in *size,* yes. [Clare reached out for yet another box, a large cereal packet, which she held alongside the small cereal packet.] Would you say that *those* two are different shapes?

R: They're similar.

Cl: What does similar mean?

R: Same shape, different sizes.

[During the last four exchanges there was hesitancy, a lot of eye contact, giggles, each person looking at others in the group, the teacher seeming to monitor the energy in the group.]

Cl: Same shape but different sizes. That's going around in circles isn't it?—We still don't know what you mean by *shape.* What do you mean by shape?

[She gathers three objects, the two cereal packets and the meter ruler. She places the rule alongside the small cereal packet.]

Cl: This and this are different shapes, but they're both cuboids.

[She now puts the cereal packets side by side.]

Cl: This and this are the same shape and different sizes. What makes them the *same* shape?

[One girl refers to a scaled-down version. Another to measuring the sides—to see if they're in the same ratio. Claire picks up their words and emphasises them.]

Cl: Right. So it's about *ratio* and about *scale.* (pp. 74–75)

Runesson suggests that the mathematical challenge presented to these students involves the discernment of similarities and differences between the packages to determine what 'the same shape' means? To assist students in this task, the teacher brings out the similarities and differences between various 3D objects:

Initially, the teacher kept the number of sides of the shapes constant, whereas the lengths of the sides varied; the lengths, breadths and the heights were not proportional ... the opening of a space for variation enabled two learners to discern those particular aspects of the object of learning. (pp. 76–77)

From Runesson (2005)

By contrasting the task focus of two lessons, Watson (2003) also illustrates how task variation can impact on students' opportunities to engage in mathematical thinking and learning. In the first lesson (see Groves & Doig, 2002), the mathematics focus was the 'concept of a circle'. The lesson began with Mr J. producing a pole for a game of quoits. The students stood along a straight line and threw rings over the pole, which was a short distance from the line. The class then discussed how easy or hard it was to do this and 'measured' and compared distances, using a tape. In response to the teacher's challenge to "make the game fair" the students conjectured that they should stand in a curve. In a second lesson at a different school, the teacher identified the lesson topic as "The chance of winning one million dollars". The activity involved the students tossing three coins and recording their results. The plenary session consisted of a rather inconclusive discussion about the different results found by the students.

According to Watson's (2003) analysis, the dimension of task variation involved in the first lesson was the distance from the point. This was enacted by students, represented by string, and made the focus for discussion. Groves and Doig (2002) reported that this focus on the mathematical properties of a circle—supported by appropriate attention to social and mathematical norms—appeared to be obvious to all the students. By way of contrast, the task in the second lesson included variations in numbers of trials, methods of throwing coins, ways of recording, and ways of comparing results. While it is likely that any teacher observing this lesson would know that the lesson was about probability, Groves and Doig noted that the mathematical focus would be less obvious for students. Thus, Watson contends that while the second lesson created multiple opportunities to learn, some students would be learning about ways of recording, others about fractions, others about how to work together, others about how to avoid working, and so on, only a few might learn about experimental probability. To increase the effectiveness of this lesson for all students, Watson suggests that as a teacher she would need to:

> narrow the range of what it is possible to learn, to discern as varying, in what I offered disparate students and thus increase the opportunity to learn appropriate mathematics for as many students as possible, while making sure that they all had access to the patterns under consideration. (Watson, 2003, pp. 36–37)

A critical focus of task activity is students' solution strategies (Hiebert et al., 1997). The *Effective Teachers of Numeracy* study (Askew et al., 1997) notes that, in addition to valuing mathematical structure, *connectionist* teachers in this project consistently value children's solution strategies. The importance of teachers noticing and attending to the mathematics inherent in their students' solution strategies is highlighted in numerous classroom studies (e.g., Carpenter et al., 1997; Sherin, 2001) and has been discussed in more depth in the preceding chapter. The following episode from a Japanese classroom illustrates how alignment of the mathematical focus of a task with the associated student activity can be effective in developing students' mathematical thinking and understanding.

Adding It Up

In the Year 1 lesson the children's problem for the day was to find the answer to 8 + 6 and explain the reasons for their answers.

Children worked individually for 5 minutes, after which the teacher wrote 8 + 6 = 14 on the board and invited particular children to write their solutions on the board.

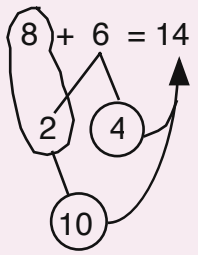


Fig. 5.4. Girl 1's solution for 8 + 6 = 14

Girl 1's solution is shown in figure 5.4. When asked, most children stated that they had used the same method. The teacher then asked the children to guess why Girl 1 had divided the 6 into 2 and 4. Children responded that this was based on "Nishimoto-san's making 10 rule"—apparently formulated by one of the children, Nishimoto-san, in the previous lesson where the problem was to find 9 + 6.

The teacher then asked for a different solution. Boy 1's solution is shown in figure 5.5:

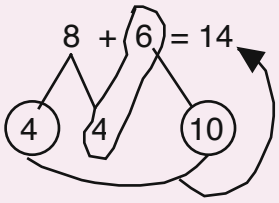


Fig. 5.5. Boy 1's solution for 8 + 6 = 14

The teacher commented that this was again using "Nishimoto-san's making 10 rule", and asked for another way. Girl 2's solution, still described by the teacher as using "Nishimoto-san's making 10 rule", is shown in figure 5.6. A few children said they had used this method.

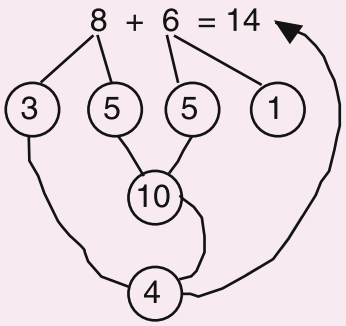


Fig. 5.6. Girl 2's solution for 8 + 6 = 14

Boy 2 stated that he did not use the "making 10 rule". Children tried to guess how he found the answer—had he used a "making 5 rule"? Boy 2 said he had not and explained his reasoning as shown in figure 5.7:

$$8 + 6 = 14$$

because $9 + 6 = 15$ "we did this before"

and 8 is one less than 9. So, "if 9 becomes 8, the answer is one less".

$$\begin{array}{rl} & 8 + 6 = 14 \\ \text{1 less} & 9 + 6 = 15 \quad \text{1 less} \end{array}$$

Fig. 5.7. Boy 2's solution for 8 + 6 = 14

Many children clapped in response to this solution and a girl commented that this used their former knowledge of addition. The teacher suggested that they move on to looking at 7 + 6 using the same method [and the lesson continued with students conjecturing and testing hypotheses around the 'making 10' rule].

It is evident that the core task focus—the mathematical structure of the numbers (e.g., 8 = 3 + 5, 6 = 5 + 1)—was enhanced by students' conjectures and justifications. The teacher's frequent use of students' solutions, both correct and incorrect, was a feature of the lesson. The lesson purpose was achieved through both the use of a task that was genuinely problematic, yet accessible, for students, and through the establishment of social norms that valued individual and group contributions to the solution process.

From Groves and Doig (2004)

As we have seen in chapter 4, tasks that pivot around mathematical negotiation and sense making afford opportunities for student engagement in mathematical practices. For the teacher, utilisation of mathematics tasks that focus on sense making require them to support students explicitly to take a more responsible role for making mathematical meaning (Anthony & Hunter, 2005; Pape, Bell, & Yetkin, 2003). In classrooms where students actively negotiate meaning, Watson (2003) has found that teachers:

> structure the context of such negotiation with examples, counter-examples, or by encouraging the development of these, so that what is eventually learnt is coherent and valid. [In addition] the teacher needs to structure the negotiation process itself so that it is mathematical … based on exemplification, generalisation, conjecture, justification and so on. (p. 32)

The role of examples

Task focus on mathematics is exemplified by the commonly utilised *worked example*. But far from being a show and tell—followed with practice by replication—the role of examples can be usefully associated with the important process of generality. Watson and Mason (2005) argue that a central aspect of learning mathematics involves "becoming familiar with examples that manifest and illustrate mathematical ideas and by constructing generalisations from examples" (p. 2). Effective instruction should, according to Watson and Mason, include collections of examples—'example spaces'—that exemplify structural similarities and differences. In the following analysis of a task and associated student activity observed in a UK numeracy lesson, Askew (2003) links the students' difficulty in discerning number properties to the inadequate provision of an example space.

Videotapes

The observed task was: *Mrs Chang bought some videotapes. She bought five tapes each costing the same amount. She spent £35. How much did each tape cost?* The essence of this problem is 'what number do I have to multiply by 5 to get 35?' Symbolically: ? x 5 = 35.[4]

This problem is difficult to represent in the physical world (using a 'model for') and several of the children used a trial-and-error approach. Using vertical mathematising (that is, working with

the symbols), the model can be recast as 5 x ? = 35. The students did not do this. Although they might 'know' that ? x 5 = 5 x ?, they appeared to find it difficult to 'uncouple' from the real-world context and move around the mathematical world instead; to move from a 'model for' to a 'model of' (Gravemeijer, 1994).

Askew argues that task selection should focus on the development of analogical thinking, enabling students to think about the structure of problems. "Rather than treat each problem afresh, the experienced problem solver has knowledge of a wealth of problems, some of which provide generic 'archetypes' that can be used to decide what category of problems a specific example fits into" (p. 84). In this case, the problem could usefully be paired with the problem: *Mr Chang bought some video tapes. He bought some tapes costing £7 each. He spent £42. How many tapes did he buy?* By exploring why one or the other may be more easily solved, children's insights into the nature of problem could be deepened.

From Askew (2003)

The work of the Cognitively Guided Instruction Project (Carpenter et al., 1999) is based on a problem-structuring framework. For example, simple addition problems can be categorised either as *change* (a given quantity is increased) or *combined* (two separate quantities are brought together). Both these categories can give rise to further categories of problems, depending on the position of the 'unknown' in the 'story'. Introducing children to the language of 'change', 'combine' and so on may help them to establish a generic set of problems, especially if these are accompanied by contextual images that can form the basis of archetypal problems (Carpenter et al., 1996).

Student-generated examples and questions can form a productive focus for student enquiry (Watson & Mason, 2005). By engaging in a public discussion of student-generated methods based on student-generated examples, the teacher in this vignette provides opportunities for her learners to become more resourceful and flexible.

Farah's Distributive Law

Farah, an experienced and highly qualified teacher, was working with her class of 11-year-olds on their understanding of multiplication. They had been assessed as having average attainment in their previous schools. In United Kingdom parlance they were classified as *middle ability*, which means that on tests they tended to come somewhere in the middle range.

She had been explicit about the use of the distributive law to deal with different place values of the digits used in large numbers. For example, they had been taught, or reminded, through calculator exploration, that 7 x 65 was the same as 7 x 60 + 7 x 5 or 7 x 50 + 7 x 5 + 7 x 10 or other such representations. Learners were asked to contribute examples of multidigit multiplications and show the whole class how they would calculate them. Some learners used a traditional approach of dealing with separate digits, such as 37 x 9 = 30 x 9 + 7 x 9, but others used ad hoc decompositions that suited the specific numbers being multiplied, such as 37 x 9 = 40 x 9 – 3 x 9.

Farah particularly praised these decompositions. Her aim was for learners to develop both flexible and mental methods for multiplication, as well as to understand distributivity. The examples were shared within the class. Two weeks later she repeated the exercise. There was a significant increase in most learners' use of flexible approaches based on characteristics of the numbers involved, rather than just separating the digits, although there had been no work on this area of mathematics meanwhile. Farah took this to be a sign that some learning had taken place as a result of the emphasis she had placed on interesting decompositions in the earlier lesson.

From Watson and Mason (2005)

In this episode, the development of exemplification is not explicitly forced by teacher-imposed constraints, but by sharing, making other possibilities available, and publicly valuing examples of what the teacher hopes others will be able to do later. By valuing a range of ways of seeing

the mathematics, the teacher encourages student creativity in arithmetic. Watson and Mason claim that this practice shifts the responsibility for learning to the learners, thus helping them to gain ownership of their mathematics.

> Asking learners to exemplify aspects of what they have studied encourages them to search through the structure from varying points of view, using a new dimension, and hence see, perhaps, for the first time, what might be there by discerning features and aspects. Thus, learners might find that being asked to exemplify gives them an opportunity to search in unfamiliar ways through what is familiar to get a more complex sense of the range of possibilities in the topics studied. (p. 31)

In a relatively small New Zealand study, Klymchuk (2005) reports improvement in student performance as a result of the explicit use of counter-examples when teaching calculus. Performance improved on questions requiring conceptual understanding but not on questions requiring the application of familiar rules, algorithms, and calculations. Watson and Mason (2005) and Carpenter, Franke, and Levi (2003) note that student-generated counter-examples are natural ways for all learners to argue and explain their mathematical thinking.

This discussion on the use of examples highlights the potential of a task format that is traditionally taken for granted within the mathematics classroom. In arguing for greater use of examples in mathematics lessons, Watson and Mason (2005) go so far as to say "that until you can construct your own examples, both generic and extreme, you do not fully appreciate a concept" (p. 32). Further research within the New Zealand context and at all levels is needed to substantiate their claim that "learning is greatly enhanced when learners are stimulated to construct their own examples" (p. 32).

Open-ended tasks: modelling, investigating, and playing

Another form of task that supports student exploration and thinking is the open-ended task. Tasks or problems are deemed open-ended when learners must engage in additional problem definition and formulation in order to proceed[5]. This 'openness' allows for a range of 'correct' responses and a range of ways of achieving those responses. Moreover, the openness of the activity fosters some of the more important aspects of learning mathematics, specifically, "investigating, creating, problematising, mathematising, communicating and thinking" (Sullivan, Warren, & White, 1999, p. 250). Open-ended tasks provide the ideal opportunity for 'mathematical play'—"that part of the process used to solve mathematical problems, which involves both experimentation and creativity to generate ideas, and using the formal rules of mathematics to follow any ideas to some sort of a conclusion" (Holton, Ahmed, Williams, & Hill, 2001, p. 403).

In addition to providing learners with the opportunity to engage in a range of mathematical practices, Zevenbergen (2001) notes the pedagogical advantages of such activity. Her study, which involved 114 primary school students, highlights how the diversity of responses and range of representations offered by open-ended problems allow greater scope for (a) teachers to assess student understanding and (b) students to demonstrate what they know. The students in this study were exploring 'average', using examples such as: *At the Chevron Island Bridge, the average number of people per car is 2.5. Draw what this might look like if there are 16 cars on the bridge.* Importantly, Zevenbergen's study highlights potential drawbacks for some students. When solving the task: *My dog weighs about 20 kilograms. How much could she weigh?,* several students struggled with the ambiguity of the word 'could': some used a futures perspective, and others, a rounding context. In a subsequent study, *Overcoming Structural Barriers to Mathematics Learning,* Zevenbergen and colleagues found that teachers needed to be more explicit about task purposes and 'rules', especially where the language may be interpreted in different ways. We revisit this in the later section on task context.

Several research studies have focused on the role of mathematical modelling activities

involving authentic problem situations, opportunities for model exploration and application, and multifaceted end products (Lesh & Doerr, 2003). In contrast to the traditional focus on arithmetic word problems, English (2004) argues that we "need to design experiences that develop a broad range of future-oriented mathematical abilities and processes. Mathematical modelling, which has traditionally been reserved for the secondary schools, serves as a powerful vehicle for addressing this need" (p. 207). The following vignette illustrates children's engagement in a mathematical modelling activity.

What Car to Buy?

The sixth grade class teacher introduced the following modelling activity in a whole class format:

Carl and his mother have been out shopping for cars. *Carl wants a car that will be fun to drive around in, gets good gas mileage, but doesn't cost too much. But Carl's mother, who is going to help pay for the car, wants him to have a car that is reliable and safe. Your job is to create a list for Carl and a list for his mother showing which cars are the best. Then they will have to decide which one to buy!*

Table of information (abridged from nine entries):

Car Style	Year	Cost	Color	Mileage	L/100 km city	Features	Body
Nissan Silva	1992	10,000	Navy blue	96,000	10	Rear spoiler, power windows, power steering, CD player, alloy wheels, alarm	Coupe
Honda Legend	1993	17,200	Dark green	15,400	12.5	Dual airbags, anti-lock brakes, alarm, cruise control, electric sunroof, power steering, power windows	Sedan

The children worked in small groups to solve the problem. The following episode is from Jasmine's group:

Charlotte: I think we should do a process of elimination. (A brief discussion ensued regarding the mother wanting the car to be reliable.)

Jasmine ... Maybe first, maybe we should do a process of elimination, so work our way down the list or work our way up.

Rachel: We have to consider all the factors though.

Douglas: (Reminding the group that they need to be objective) She (the mother) is helping him. How about we just judge off what the thing says, not by what we think ... Let's all read through this again and underline all the details that help us.

After a number of minutes spent revisiting the goal and re-interpreting the problem information the group returned to the process of elimination, with considerable argumentation around the suggestion that the most expensive car be eliminated first. Expectations to justify claims were common: "Jasmine, tell us why you think these things. We need to know why you think them."

The conversation continued with the group reverting to argumentation over cost factors versus leisure features of the cars. Again Douglas reiterated the need to be objective: "we're not deciding on what we like, we're deciding on the facts ... we have to look at these factors." Finally the group devised a rating system based on the most important features: safety, leisure and extras, mileage.

Car Preferences for Carl and his Mum (abridged from nine entries):

Car	Carl	Mother	Total
Nissan Silva	4th	4th	4th
Honda Legend	7th	6th	7th

At the end of the activity, each group of children shared with the class their approaches to working the activities, explained and justified the model they had developed, and then invited feedback from

their peers. The group report was followed by a whole-class discussion that compared the features of the mathematical models produced by the various groups.

The problem provided rich opportunities for the group to engage in a range of mathematical practices, along with the development of important mathematical constructs of ranking, weighting ranks and selecting and aggregating ranked quantities that are embedded within the problem. The mathematical practices included interpreting and re-interpreting the problem information, making appropriate decisions, justifying one's reasoning, posing hypotheses and problems, presenting arguments and counter-arguments, applying previous learning, and acting metacognitively.

From English (2004)

In a three-year longitudinal study of elementary children's development of mathematical modelling, Watters, English, and Mahoney (2004) demonstrated how engagement in extended modelling problems provides opportunities for learners to engage in a range of mathematical processes and develop mathematical understanding. Because the modelling activities in the study were designed for small-group work, they also provided useful opportunities for developing collaborative problem-solving skills. In addition, children developed important metacognitive and critical thinking skills that enabled them to distinguish between personal and task knowledge and to know when and how to apply each during problem solution. Tanner and Jones (2002) worked with six secondary schools in Wales to investigate the use of mathematical modelling tasks with 24 classes of eleven- and twelve-year-old students. Using a matched quasi-experiment with pre- and post-tests and accompanying classroom observations and interviews, the researchers reported positive but small effect sizes when teaching interventions involved the explicit integration of metacognitive thinking skills such as planning, monitoring, and evaluating with the practical modelling activities.

As noted earlier, and in the Early Years chapter, learner-generated examples, interests, and questions provide a rich source of investigative-type activities for learners and, additionally, important feedback for teachers. The following vignette from Biddulph's (1996) research investigating the nature of questions that young learners ask about geometry illustrates how student questions can give insight into their thinking and provide a useful starting point for investigations.

Children's Questions

Biddulph (1996), in a small study involving 100 children aged eight to eleven years, noted that student-generated questions revealed considerable insight into their thinking and understanding. With respect to geometry, a proportion of the children's questions suggested that the children already had some understanding of particular concepts, for example:

- "How can you find the line of symmetry?"
- "Do you always need to draw squares of a grid to enlarge something?"

Other questions, however, revealed considerable lack of understanding, particularly with reference to angles:

- "Do angles cross?"
- "Are angles something that lean to one side?"

Biddulph noted that of the 73 questions in geometry, approximately 80% could provide the basis of worthwhile investigations. For example, the following questions about tessellations could be investigated together:

- "What shapes can be in a tessellation?"
- "Can you use two shapes to pave a driveway?"
- "Can it work with a shape with wavy lines?"
- "Does it change if the shape is different sizes?"

The children's questions in Biddulph's study also included some that illustrated that children's feelings are integral to their learning of mathematics. For instance, one child asked in exasperation, "I wouldn't have clue how to do it; what's the use of this?" Another child was concerned to know, "Is geometry safe?"

From Biddulph (1996)

Mathematics teaching for diverse learners ensures that tasks link to learners' prior knowledge and experiences

> The learning task is its conceptual component; the learning activity is the task's practical counterpart, or the means through which the teacher intends the child to make the required conceptual advance from what was learned previously to what must be learned now. (Alexander, 2000, p. 351)

An important factor in the implementation of any task is the relationship between learner and task (Turner & Meyer, 2004). Tasks that provide students with opportunities for success, present an appropriate level of challenge and difficulty, and increase students' sense of control and arouse their interest can help elicit intrinsic motivation.

> When students engage in tasks in which they are motivated intrinsically they tend to exhibit a number of pedagogically desirable behaviours including increased time on task, persistence in the face of failure, more elaborative processing, the monitoring of comprehension, and selection of more difficult tasks, greater creativity and risk taking, selection of deeper and more efficient performance and learning strategies, and choice of activity in the absence of extrinsic reward. (Middleton & Spanias, 1999, p. 66)

Conversely, if tasks are inappropriate in terms of motivation, interest value, or learners' prior knowledge—or simply because they lack suitably specific task expectations—student engagement may well be lower than anticipated or desired (Stein et al., 1996).

Planning task activities and learning goals

Planning appropriate learning sequences is an essential role for teachers (Cobb & McClain, 2001). Supported by research-based frameworks, there has been a move in recent times to design tasks around learning trajectories that signal important signposts for student learning (e.g., stages in the NDP Number Framework or 'growth points' in numeracy (Clarke, 2001)), and a substantiated means of supporting and organising this learning. Evaluation reports of the NDP and Te Poutama Tau consistently report that the Number Framework and associated Strategy Teaching Model have helped create for teachers a series of reference points for planning 'where to next?' tasks to meet individual needs (Higgins et al., 2004; Trinick, 2005).

Learning trajectories provide a general overview of the learning continua of the classroom community, not of individual students: students do not all progress along a common developmental path (van den Heuvel-Panhuizen, 2001; Wright, 1998). McChesney (2004), in a study of three New Zealand year 9 and 10 classes, found that student development of number sense was a nested and recursive process. Because students did not move through the stages of the Number Framework in a linear fashion, it was important that classroom activity "afforded opportunities for students to return to familiar mathematical entities, and negotiate further mathematical meanings" (p. 301).

When designing and implementing instructional tasks, it is important that learning goals and activities are adapted in response to teachers' perceptions of students' levels of understanding and in response to their ongoing evaluation of students' performance. Within the context of the objective-based numeracy lesson commonly found in English classrooms, Askew and Millett (in press) stress the need for flexibility that allows teachers to be "both responsive to students

and responsive to the discipline" (Ball, 1997). Their research found that in many instances the transformation of lesson objectives into meaningful learning experiences was mediated by teachers' knowledge of the subject matter. To illustrate, Askew (2004b) provides an analysis of a lesson based on the stated objective, "to understand multiplication as repeated addition." Based on observation, it was clear that the teacher's intention was that children should use addition as a means of *calculating* multiplications. In the teacher's account of the lesson, she emphasised that this meant that children have strategies that they understand. However, rather than building an understanding that multiplication is more efficient than addition, Askew concluded that some students could be building an understanding that addition is the foundation of multiplication. This could potentially lead to difficulties with understanding calculations such as ½ x 12.[6] The interpretation that 'you can use addition to find answers to multiplication calculations' was observed by Askew and Millet in several other numeracy classrooms.

Quality teaching based on appropriate sequencing also requires flexibility of task implementation. A concern raised by Askew and Millett (in press) relates to the flexibility with which teachers are able to work with students' methods or the methods pre-specified in lesson objectives. Objectives that are too small (e.g., a learning intention 'To add or subtract the nearest multiple of 10, then adjust') may lose sight of the big picture. Askew and Millett's account of a numeracy lesson exemplifies how a teacher, focused on helping children carry out a specific calculation strategy, can fail to help them make meaning from the tasks by not locating them within a broader network of mathematical ideas.

Managing learning trajectories through tasks that allow multiple entry points is the focus of an ongoing research project, Maximising Success in Mathematics for Disadvantaged Students (Mousley, Sullivan, & Zevenbergen, 2004). In their earlier project, Overcoming Structural Barriers to Mathematics Learning, these researchers found that the practice of having all students complete the same open-ended task, with the only difference being in expectations for performance levels, failed to recognise the needs of diverse learners. In an attempt to counter potential negative effects of differentiated learning *expectations,* the current three-year project examines an alternative practice of task differentiation in which the whole class completes the same basic task. Students experiencing difficulty are provided with alternative pathways, shaped by 'prompt' activities that provide stepping stones leading to the learning goal:

> We suggest that a sense of community is more likely to result from teachers offering prompts to allow students experiencing difficulty to engage in experiences related to the initial goal task, rather than, for example, requiring such students to listen to additional explanations, or to assume that they will pursue goals substantially different from the rest of the class. (Sullivan et al., 2004, p. 259)

Within the study, students who finished a task quickly were posed supplementary tasks that extended their time on that task, rather than proceeding onto the next stage in the lesson. This approach was successfully trialled across several classes. Adjustment of the task, based on students' responses, enabled all students to participate in class discussion and reviews and, most importantly, all students were prepared to move together on to the next stage of the learning. While the trial lessons found that this form of task differentiation changed the learning experience of students, especially for those who were at risk of not being able to follow learning trajectories set with the whole class in mind, the researchers noted the need to examine large-scale implementation of this approach, along with other forms of task differentiation.

Bicknell (2003) provides an example of an open-ended task suitable for whole-class work. Her account of a year 3–4 teaching unit on volume, based around the packing of boxes, shows how students were able to work at levels appropriate to their developmental stage. The research identified clearly distinguishable levels related to the children's experience and proficiency with number strategies. Some children in the class moved easily from a model 'of' the packing situation towards a model 'for', generalising their strategies so that they applied to the volume of a rectangular prism.

Flexibility in task sequencing requires that teachers themselves engage deeply with the concepts involved. This engagement can occur both during planning and during teaching. In examining the planning process through a case study of one teacher, Mary, it emerged that lesson planning was as much a process of learning as it was a process of preparing for the contingent activity of teaching. Rosebery (2005) found that Mary conceptualised her own learning in much the same terms as those she used when thinking about her students' learning:

> She expects to examine what she knows from a variety of perspectives, including past and present experience, 'official' meanings (e.g., from teachers' manuals, textbooks, [and mathematics advisor] and her students' understanding. ... Mary is not satisfied with her own understanding until she has integrated these potentially disparate points of view into a coherent whole. Nor is she ready to make decisions about teaching until she has achieved this. Thus for Mary, lesson planning is as much a process of learning as it is of teaching. (p. 323-4)

Teachers' report that supporting material in numeracy professional development projects (e.g., the New Zealand *Numeracy Development Project* and England's *National Numeracy Framework*) provides useful guides for knowing 'where to next', and being clear about 'what's expected' (Askew et al., in press; Higgins et al., 2005). Likewise, teacher development studies in the US report the benefit of increased teacher knowledge of students' thinking related to frameworks or progressions (e.g., Carpenter et al., 1997; Doerr & Lesh, 2003). In a professional development programme focused on typical milestones and trajectories of children's reasoning about space and geometry, Jacobson and Lehrer (2000) found that in those classes in which teachers were more knowledgeable, not only did students learn more than their counterparts, but this difference in learning was maintained over time.

Connecting tasks to learners' existing proficiencies and knowledge

Linking mathematics tasks to learners' existing proficiencies is a significant factor in the maintenance of high-level cognitive activity (Stein et al., 1996). Productive task engagement requires that tasks relate sufficiently closely to students' current knowledge and skills to be assimilated, yet be different enough to transform their methods of thinking and working (Grugnetti & Jaquet, 1996).

Fennema et al.'s (1996) longitudinal study of 18 teachers from the Cognitively Guided Instruction Project found achievement in concepts and problem solving was higher when instruction was designed around students' existing proficiencies and concept images. This approach replaced the more traditional approach where teachers focused on filling gaps in students' knowledge or remediating weaknesses. Likewise, increased performance gains recorded in the New Zealand NDP are, in part, attributed to teachers' increased awareness of their children's existing knowledge and the consequent focusing of instruction on "where students are at" (Higgins, 2003).

Understanding 'where learners are at' is not, however, always easy. In order to understand students' current thinking, teachers need to be able to deconstruct their own knowledge into a less polished, less final form—to work backwards from a mature and compressed understanding of the content in order to unpack its constituent elements (Ball, 2000). The following vignette illustrates how teachers and students can have different interpretations of the mathematics under discussion.

Dwindling Savings

Seventh grade students are discussing the expression 15000 – 300w for calculating somebody's dwindling savings as a function of the number of weeks (w) during which the money was spent.

Teacher:	Would anyone do anything differently? Martha?
Martha:	I'd do 15000 minus brackets, 30 and number of weeks [writes: 15000 – (300 w)].

Teacher:	All right. Do we need brackets around this? [points to 300 w]
Simon:	Yes, you do, because you have to know that there's an operation. A person, now, he'll probably think *300 weeks*, not *300 times* weeks.
Teacher:	OK, anyone who now knows algebra will know there is an operation.

From the exchange it appears that the teacher was unaware that the algebraic expression may have been initially interpreted by the students as an abbreviated sentence in which letters, such as w, were a shorthand for nouns, such as weeks, rather than placeholders for numbers.

From Sfard (2005)[7]

The episode also illustrates that, in any learning situation, children create their own meanings—meanings that sometimes form useful building blocks and at other times are not appropriate at all. Here 'appropriateness' refers not to the inner coherence of students' knowledge but to possible disparities between students' conceptions and the public versions of the same idea (Davis, 1998). For example, many young children quite sensibly transfer the belief that multiplication makes bigger and division smaller to fractions, at least in the first instance. As such, it is important to regard the appearance of misconceptions as a natural part of the learning process rather than a contradiction of a vision of learners as sense-makers:

> Indeed, not in spite of students' need for meaning but rather because of it, students tend to construct their own conceptions. Precisely because of their need to fit new concepts into their former knowledge, their understandings are sometimes at odds with the official definitions. (Sfard, 2003, p. 357)

Teachers who understand their students' thinking as 'understanding in progress' are typically able to use such misconceptions as building blocks (see Groves and Doig, 2004), creating tasks that challenge misconceptions and provide opportunities to resolve cognitive conflict.

> The presentation of a possible case for conflict is much more than an attempt to enculturate the student into conventional understandings, or to extend the student's experience of a particular discourse, it is an induction into an internal validity of mathematics by offering the student an opportunity to rethink and restructure existing assumptions and understanding, or at the very least to realise that there is a problem. (Watson, 2003, p. 31)

Watson illustrates this with an episode involving a student who thinks that the derivative of x^2 is $2x + k$. To continue the negotiation, the teacher asks for the derivative of $x^2 + kx$. By offering a very particular example, the teacher informs the negotiation process—not by judging the student's ability but by giving the student the responsibility for sorting it out for herself.

In order to link tasks to students' thinking and experiences and to make appropriate choices regarding the difficulty level and degree of task explicitness, teachers need detailed knowledge of their students. While such knowledge can be built up from a range of assessment practices, listening to students' mathematical explanations and justifications is seen as a necessary component (Davis, 1997). Teachers, in listening, need to notice significant mathematical moments and respond appropriately. This is a complex task (Sherin, 2001). Watson (2000), in a UK study of 30 primary, middle, and secondary teachers' informal assessment practices, expressed concern about the value of the 'explain how you did this' approach. Students' oral contributions more often reported mathematics already 'done', thus limiting teacher access to how their students are thinking mathematically. A recent New Zealand study (Davies & Walker, 2005) reports on the use of classroom videotapes to focus teachers' attention on the mathematics in classroom interactions. Teachers reported that increased awareness of significant mathematical moments meant that they were more likely to respond to the mathematical learning needs of their students. As one teacher pointed out:

> Something that I've learned this year is that you can let the children guide a lot of the learning. It's OK to stop and smell the roses a little bit. If something comes up, that's a great little teaching moment—go for it—grab it—even it if goes off

> on a tangent somewhere, rather than having to just stick to the book—close the book—someone's brought up that idea so let's talk about that. (p. 278)

Frameworks, such as the Number Framework in the NDP (Ministry of Education, 2006), provide a basis for understanding critical stages in children's development. When used in conjunction with a diagnostic interview focused on students' understanding and solution processes, frameworks can usefully map children's thinking (Clarke 2001; Irwin & Niederer, 2002; Thomas & Tagg, 2004).

> The assessment interview has given focus to my teaching. Constantly at the back of my mind I have the growth points there and I have a clear idea of where I'm heading and can match activities to the needs of the children. But I also try to make it challenging enough to make them stretch. (Clarke, 2001, p. 20)

Teachers involved in numeracy professional development (e.g., in New Zealand, Australia, and England) reported being surprised (at least in the initial interviews) at what many children were or were not able to do:

> In every class there is that quiet child you feel that you never really 'know'—the one that some days you're never really sure that you have spoken to. To interact one-to-one and really 'talk' to them showed great insight into what kind of child they are and how they think. (Clarke, 2001, p. 14)

Teachers in all of these numeracy projects report that their use of diagnostic interviewing has helped them move to a more responsive pedagogical practice that uses students' existing mathematical knowledge as a basis for task design and instructional direction.

Situating tasks in contexts

Across all curricula, opportunities to explore authentic applications that arise out of real-life contexts can have a significant and sustained impact on student knowledge, attitude, self-esteem, independence, and confidence (Alton-Lee, 2003). In order to make mathematics more meaningful and accessible for all learners, mathematics curricula frequently advocate the use of contexts. In this sense, 'context' refers to a real or imaginary setting for a mathematical problem, which illustrates the way the mathematics is used.

Advocates claim that the use of contexts can motivate, illustrate potential applications, provide a source of opportunities for mathematical reasoning and thinking, and anchor student understanding (Meyer, Dekker, Querelle, & Reys, 2001). For example, Wiest's (2001) study of 273 year 4 and 6 children's problem solving found that the selected contexts affected a range of variables, including the children's interest in, attentiveness to, and willingness to engage with problems, the strategies they used, their effort, their perception of and actual success, and the extent to which they learned the intended mathematics. For these young children, fantasy contexts evoked stronger responses than other contexts. Wiest argues that fantasy contexts offer an appealing link to abstract and creative thinking but notes that, as with any 'contextual' motivator, it is more important to seek long-term benefits than short-term 'feel-good factors'. Moving from fantasy to reality, several research studies (e.g., Watson, 2000) have found newspaper reports to be a useful source of contextual problems. Arnold (2005) provides examples of calculus problems related to ozone depletion and IBM share prices that he has used with senior students, claiming that there is much value in asking students to think critically about what they read in the news.

Internationally, the most prominent advocate for contextually-based tasks is the Dutch Realistic Mathematics Education (RME) movement. In this programme, Gravemeijer (1997a) maintains that the primary use of context is not to motivate students but to provide a learning situation that is experientially real for the students[8] and which can be used as a starting point for advancing understanding. "What is important is that the task context is suitable for mathematisation—the students are able to imagine the situation or event so that they can make use of their own experiences and knowledge" (van den Heuvel-Panhuizen, 2005, p. 3) to support the development

of context-related strategies and notations that later become more generalised. For example, when solving problems such as dividing three pizzas among four children, students arrived at solution methods that pre-empt the more formal procedures for adding and subtracting fractions with unequal denominators (Streefland, 1991).

Watson (2004) also argues that 'realistic' does not mean that tasks must *necessarily* involve real contexts. Citing a student of Goldenberg (1996), she advocates that tasks should be seen as 'realistic' not because they relate to any particular everyday context, but because they make students think in 'real' ways. Watson noted that students in the Improving Attainment in Mathematics Project were usefully motivated and intrigued by tasks that exemplified the 'power' of mathematics. Similarly, Ainley and colleagues (2005) claim that the purposeful nature of activity is a key feature of relevance:

> This use of 'purpose' is quite specifically related to the perceptions of the learner. It may be quite distinct from any objectives identified by the teacher, and does not depend on any apparent connection to a 'real world' context. (Ainley, Bills, & Wilson, 2005, p. 18)

Purposeful tasks provide students with opportunities to use and learn about particular mathematical ideas in ways that allow them to appreciate their utility. Ainley et al. (2005) provide an example of students solving a 'pyramid' task in the form of a contextualised game. Students who had an appreciation of the purpose of the exploration, which was to find *how* to get the highest total (as opposed to *calculate* the highest total) were more likely to attend to the underlying mathematical structure of the array output in the spreadsheet. In addition, some students were able to appreciate and articulate the utility of standard notation (in this case, Excel formulas) for clarifying the ways in which the total was calculated.

Research findings (e.g., Burton, 1996; Salomon and Perkins, 1998) demonstrate that the use of context in mathematics tasks does not automatically lead to improved learning outcomes for all students. Sullivan et al. (2002) argue that problem contextualisation may actually contribute to the disempowerment of some students. Certain contexts that are taken as realistic and relevant may in fact restrict the mathematical development of students from socially and culturally divergent backgrounds.

Lubienski (2000) found that contextualising tasks obscured their mathematical purpose for some students. In particular, students categorised as lower SES tended to focus on the contextual issues of a problem at the expense of the mathematical focus and treated problems individually without seeing the mathematical ideas connecting them. Likewise, Cooper and Dunne's (2000) review of the national testing system in the UK found that contexts created another layer of difficulty for some students—particularly those from working class backgrounds who appeared to approach the problems in heavily context-laden ways unintended by test writers. In New Zealand, Forbes (2000) raised similar concerns. In her study on assessment, Māori students appeared to be disadvantaged by the contextual information in the tasks. The tangible, 'concrete' context expected to assist students was not necessarily concrete in the sense of making sense. Meaney and Irwin (2005), in their investigation of four tasks from the National Education Monitoring Project (NEMP), found that year 8 students were far more successful at recognising the need to 'peel away' the story shell of the problems. A year 4 response to the request to work our how many cars went down a motorway in 9 minutes if 98 cars went down every minute was: "I think 259 … because, umm, there's lots of car going up and down, and um, cars like going visiting and on the bridges." In another investigation involving NEMP tasks, Anthony and Walshaw (2002) noted a similar trend, with year 4 students more frequently accessing their informal knowledge. In their attempts to solve problems, students' real-world concerns sometimes collided with the mathematical solution: When asked to describe "How much of the pizza is left?" a year 4 student responded, "All the herbs."

Rather than abandon the use of contextual tasks, Sullivan, Zevenbergen, and Mousley (2002) argue for more critical task selection and implementation. Tasks that simplify real situations

unrealistically (see discussion on word problems in van den Heuvel-Panhuizen, 2005) or use mathematics to solve problems in unrealistic ways (e.g., working out the number of slices in a cake from the number of decorative patterns on a slice) or use situations which would be unfamiliar to all students (see Taylor & Biddulph, 1994, for context issues in probability) should be avoided.

Deciding which contexts are familiar to students is challenging. What might appear on the surface to be suitable context is not necessarily one with which all children are familiar, as is evidenced in a study of seven- and eight-year-old New Zealand children's experiences with money (Peters, 1995). Based on interviews and role play, Peters concluded that money is not a familiar context for addition and subtraction. All the children who were able to spend money accurately did so one coin at a time or purchased one item at a time. Given that this research was completed some 10 years ago, before EFTPOS transactions were introduced, it seems even more likely that programmes that use money as a context for learning early arithmetic skills will not necessarily reflect young children's ideas or experiences about money. Brenner (1998) and Guberman's (2004) investigations of children's experiences of learning about money at home and at school (see also discussion on money activities in chapter 6) also note that differences between everyday and school mathematics can make the inclusion of money contexts problematic, especially in relation to decimal notation. Rather than discard money contexts, Brenner suggests teachers infuse the curriculum with more open-ended problems and provide opportunities for student–teacher discourse to help bridge the gap between everyday and school mathematics. In connecting everyday commonsense with the curriculum, it is essential to acknowledge that "the connection does not lie in the curricular materials but in the discussion and reasoning that can take place when such materials are used" (p. 153).

While not all mathematical tasks need originate from students' cultural experience, it is necessary that embedded contexts are accessible to all students. Embracing culturally contextualised pedagogy is not, however, simply a matter of incorporating ethnic symbols and artefacts into tasks. McKinley, Stewart, and Richards (2004) report that the superficial use of a small number of "Māori contexts" in junior secondary mathematics school programmes can lead to these cultural activities being seen as "Māori caricatures". In an effort to provide a more sound pedagogical approach, Anderson and her colleagues (2005) describe the successful implementation of tukutuku panel activities into their pre-service mathematics programme. Programme evaluation reported support for the inclusion of cultural activities; students felt confident that they could integrate these within future mathematics programmes.

Clarke (2005) advocates the role of context as a social construction:

> Context in our view is neither a neutral background for the negotiation of mathematical meanings, nor merely a catalyst mediating between tasks context and the individual's mathematical tool kit. Rather we should speak of the personal task context as an outcome of the realization of the figurative context within the broader social context. (Clarke & Helme, 1998, p. 130, cited in Clarke, 2005)

With reference to recent curriculum examples from South Africa, Clarke illustrates how the incorporation of themes of societal significance can affect students' engagement with, and perceptions of, mathematics. Analysis of student–student interactions in South African classes found that AIDS provided a genuine context that allowed the nature and purpose of the classroom activity to be constructed in socio-cultural—not just mathematical—terms. Clarke argues that dichotomisation of real-world and school mathematics was avoided by "viewing students as simultaneously members of complementary communities of practice within a broader integrative socio-cultural context" (p. 9).

Sullivan, Mousley, and Zevenbergen's research project, Overcoming Structural Barriers to Mathematics Learning, aimed to address those factors inherent in the culture of schooling that constrain some students (often those from low-SES backgrounds) from engaging in context-based tasks. Multiple case studies in primary school classrooms have examined pedagogical

strategies in which teachers make *explicit* those hidden aspects of pedagogy that can inhibit student participation in open-ended contextually-based tasks. In Sullivan et al.'s 2003 study of three grade 6 teachers teaching a graphing activity based around a worm farm context, the explicit pedagogical factors were in all three cases related to student activity and outcomes. All three teachers made it explicit that there could be multiple responses to the task, but only two emphasised to their students that they could be creative in their responses. It was observed in one class that a lack of direct reference to the practical and creative possibilities of the task, combined with an explicit emphasis on the mathematical aspect, constrained the outcomes. As one observer noted:

> On looking at the work, they seemed to have done mathematically correct descriptions of the graph, but they were not particularly creative in their interpretations. The issue of their initial lack of creativity was interesting. Perhaps they have too little scope in their knowledge of worm farms to be creative. Alternatively perhaps they did not link the creative interpretation with maths. (Sullivan et al., 2003, p. 271)

The use of everyday contexts acknowledges the fact that students of all backgrounds have commonsense understandings of mathematics "that need to be harnessed and reconciled with the more general and powerful mathematical knowledge that students learn in school" (Brenner, 1998, p. 153). Learners' real-life, circumstantial knowledge is referred to in the literature as 'intuitive knowledge' (Resnick & Singer, 1993), 'situated knowledge' (Brown, Collins, & Duguid, 1989) and 'informal knowledge' (Saxe, 1988). Several New Zealand studies (e.g., Hunter, 2002; Irwin, 2001) claim benefits for the use of everyday contexts—both to support mathematical learning from informal and prior knowledge and to challenge misconceptions. Hunter and Anthony (2003) outline a teaching approach to decimals that involves using percentages and filling containers with water. The activity provides a visible representation of decimals that connects with students' everyday experience.

Irwin's (2001) research investigated 11- and 12-year-old students' learning of decimals. Pre- and post-testing revealed that students who worked on contextual problems made significantly more progress in their knowledge of decimals than those who worked on non-contextual problems. Analysis of paired interactions suggested that greater reciprocity existed in the pairs working on the contextualised problems. Irwin suggested that this was partly because, for contextualised problems, students who traditionally were lower achievers made greater use of their everyday knowledge of decimals. In addition, Irwin claimed that requiring students to link their scientific and everyday knowledge gave them opportunities to confront a range of misconceptions such as 'one-hundredth' is the same as 0.100 or '¼ can be written as either 0.4 or 0.25'. Like other researchers, Irwin found that the benefits of contextualising are mediated by the match that contexts make with students' experience and misconceptions. As such, problems involving exchange rates (seen by students as relevant because of their parents' experience of overseas travel and the practice of sending money overseas) were selected ahead of the more typical textbook contexts involving cricket or baseball statistics. The teachers in Irwin's study noted that information on relevant contexts was readily obtained through discussion with students.

In a three-year study involving year 5 learners of English as an Additional Language (EAL), Barwell (2005) also found there were benefits to be gained by linking word problems to students' informal knowledge. Earlier research suggested that the use of word problems could prove a barrier to mathematics students with low levels of proficiency in both their native language and English (Clarkson, 1992), but Barwell found that the use of a familiar context facilitated a shared sense for the emerging word problem and its solution as well as facilitating students' social relationships.

Pocket Money

In a study spanning 3 years, Barwell (2005) explored the nature of participation of year 5 learners of English as an additional language in mathematics classroom interaction. His particular focus was on how these students made sense of arithmetic word problems. Barwell found that when mathematics is the focus students do not bring to the foreground their difficulty with the English language or their use of it. Instead they use their personal experiences of the issue under consideration in the word problems as a starting point for student discussion.

Word problems included situations around furniture stores, pocket money, McDonalds, and pizzas. The students all used their individual experiences as a general backdrop for discussions about the details of the word problem. For example, in a problem involving pocket money, the students' discussion included consideration of whether £5 is a sensible amount and how much pocket money they each received, in order to develop a shared understanding of the context of the problem. Those personal experiences and their accounts of them, operated to develop a shared sense of context between mathematical partners and helped to establish a common goal to find a mathematical solution. Those common experiences also served the purpose of creating a sensibility towards a solution's meaningfulness and its relevance. "The study highlights the close interweaving of the negotiation of participant's relationships and their work on the word problem task" (p. 345).

The provision of a shared learning experience may go some way to counter the detrimental effects created when knowledge of a problem context is not distributed evenly among students from different social backgrounds. Barwell suggests that links to personal experience may be significant for all learners but particularly significant for learners of EAL, who may not so readily have access to the discourse of school mathematics.

From Barwell (2005)

In addition to issues of relevance, the language demands of contextual problems can be a significant barrier for some students (Draper & Siebert, 2004). Recent studies (Koedinger & Nathan, 2004; Lawrence & Patterson, 2005; Zevenbergen, Hyde, & Power, 2001) have found that many senior secondary students have ineffective strategies for dealing with word problems. Many students base their interpretations on trigger words such as 'more' (suggesting an additive operation). Anthony (1996b), in a year-long study of a year 12 class, reported the frequent teacher use of keywords as an instructional strategy. For example, in a review session, the teacher made these comments:

T: When you hear 'gradient' or 'tangent', what should you think about?

T: What are the keywords—what should the words 'rate of change' tell you?

T: Look at the paper—the most important thing is to find the keywords.

While a keyword may help some students *complete* a problem, keywords do little to help students construct meaningful mathematical knowledge. By compensating for skill and knowledge deficits, keywords enable students to complete problems without necessarily understanding the situation, without modelling it mathematically, and without acquiring the intended procedural knowledge.

Situational contexts can give students access to mathematics, but research has shown that they should not be the goal of a learning experience nor should they be allowed to blur the focus of the mathematics.

Providing appropriate challenge

When selecting a task, all teachers consider the level of challenge it presents. Teachers who provide moderate challenges for their students signal high expectations, and their students report higher self-regulation and self-efficacy together with a greater inclination to seek help (Alton-Lee, 2003; Middleton, 2001). Mathematical tasks that are problematic and offer

an appropriate degree of challenge have high cognitive value. Tasks that are too easy or too hard have limited cognitive value (Henningsen & Stein, 1997; Williams, 2002). Francisco and Maher (2005) report on a longitudinal study involving a group of students, tracked from grade 1 to university, engaged in well-defined, open-ended mathematical investigations. The study involved an examination of (a) videotapes of the students' mathematical behaviour and written work in problem-solving activities, (b) student reflections on their experiences, which were collected during clinical interviews in grades 11 and 12 and via follow-up questionnaires, and (c) results of detailed analyses of the development of students' ideas and ways of reasoning. The researchers claimed that providing students with the opportunity to work on a complex task—as opposed to a series of simple tasks devolved from a complex task—was crucial for stimulating their mathematical reasoning and building durable mathematical knowledge: "The opportunity to attend to the intricacies of a complex task provides the students with the opportunity to work on unveiling complex mathematical relationships, which enhances deep mathematical understanding" (p. 371).

Research has consistently documented differences in levels of challenge in the tasks provided to students of different ability groupings. The QUASAR project (Stein et al., 1996) was based on the premise that prior failures of poor and minority students in the US were due to a lack of opportunity to participate in meaningful and challenging learning experiences rather than a lack of ability or potential. Houssart (2001) found that teachers in England tended to offer challenging tasks to 'higher' sets and use step-by-step approaches with 'lower' sets. Teachers of 'higher' sets showed more enthusiasm for investigative tasks that encouraged creativity:

> There has to be an element of challenge about it ... they want to be tested in what they're doing and not feel they're doing something babyish or below them.
>
> Challenge, especially with the top set ... Probably, had I had the lower set, the challenge bit would … be far lower down, until they got the basics in obviously.

When describing a 'good' task, many of the 'bottom set' teachers in Houssart's study talk in a way that contrasts starkly with 'top set' teachers' talk of challenge and creativity:

> They like colouring in. I always start with the concrete and what they know first, we did some little games … games are successful.

Of concern is Houssart's (2005) finding—noted as striking—that many of the low-attaining students in her study preferred more challenging tasks.

> Sometimes it was just a case of the task getting harder, at other times it was making it less routine or repetitive. … The most striking finding was that whenever a child refused to do a mathematical task, there was a possible explanation in the nature of the task, even if other factors contributed. Even those children coming to mathematics lessons, apparently unprepared to do anything eventually opted in as different tasks were offered, and there was always a possible explanation in the nature of the task itself. (p. 72)

From the Australian literature, we find similar reports. Senior teachers from two secondary schools reported the use of explanatory or investigative methods with 'able' students and 'show and tell' with 'less able' students (Norton, McRobbie, & Cooper, 2002). Only a small number of the 162 primary school teachers surveyed by Anderson (2003) indicated that *all* students could learn by doing open-ended and unfamiliar problems on a regular basis. Anderson offers this response from an experienced teacher as illustrative of the reasons why this might be so:

> It's safer—children feel more comfortable if they're not made to think. I realise this is cynical—but for many children with low IQs and poor/non existent English language skills, the concept of problem solving is alien. Also it takes up too much time and there is great pressure to "get through" the curricula. So whilst in theory I acknowledge the potential of problem solving, in reality with some clientele it's too hard. (p. 76)

Zevenbergen (2005) examined students' experiences in streamed classes in secondary schools.

Based on interviews with 96 students and 10 teachers from six quite different schools, she reported that differences in pedagogical practices related to access to mathematical opportunities as evidenced by content coverage. After completing a unit of work, the students sat a common test consisting of three levels of questions that varied in complexity and application. Students from high-streamed classes reported that their lessons covered more content than required by the test. In contrast, lower-streamed students reported:

> Jaclyn: When it comes to exams, we can't do the work and can only get low marks. Which means we have to stay in the dumb class. I get so annoyed and just want to leave it but you know, you can't do anything about it. (p. 614)
>
> Becky: I would like to be in the top classes [because] they get the good teachers and they can learn the stuff and then do well in the exams. We are lucky if we can pass. We're not idiots, but the teachers think we are. (p. 616)

These studies and others challenge pedagogical practices based on simplification and repetition for low-achieving students. From her research with low-attaining students, Watson (2002) proposed an alternative approach focused on mathematical thinking. Using tasks selected by the classroom teacher, Watson taught the class, integrating a series of prompts (see Watson & Mason, 1998) designed to encourage pattern generalisation and the use and generation of examples, communicate a sense of mathematical concept, and describe underlying structures. The low-attaining students in her study showed that they were able to develop the capacity for exemplifying, generalising, abstracting, reflecting, and working with structure and images—kinds of thinking that marked them as successful novice mathematicians.

Following on from this study, Watson and De Geest (2005) worked with teachers in the Improving Attainment in Mathematics Project (IAMP) to enhance instructional practices that would support the mathematical thinking of students previously identified as low attainers. Based on their belief that these students were entitled to access mathematics, they chose not to simplify mathematical activities. They planned tasks that encouraged links with previous learning and were responsive to students' responses. Rather than use worksheets, the teachers were likely to develop tasks from starter tasks and the students' own questions. All teachers used 'create your own example' tasks as part of their everyday lesson structure and several used 'If this is the answer, what is the question?' tasks. In class, students were given more thinking time to complete tasks.

Matching tasks to the unique learning characteristics of students is particularly important for learners with special needs. Working with second and third grade students identified as learning disabled, Behrend (2003) found that, given the opportunity, they were capable of sharing their computation strategies, listening flexibly to other children's strategies, discussing similarities and difference in strategies, justifying their thinking, and helping each other understand word problems. In addition, the students were capable of generating and utilising their own problem-solving strategies and did not need to be taught specific strategies. Similarly, Thornton, Langrall, and Jones (1997) detail classroom episodes in which elementary students with significant learning disabilities successfully engaged with rich and meaningful problem tasks. For example, students tackling the problem, *Is every triangle ½ of a rectangle? Yes or no? Prove it,* demonstrated multiple solution strategies based on the cutting and re-forming of triangles and extended the task to conjecture that every triangle is half a parallelogram. The conjecture raised questions about the defining properties of shapes—for example, when is a parallelogram a rectangle?—providing an opportunity for the teacher to follow up on the distinction between congruent shapes and shapes that have the same area.

Quality teaching provides intellectually and academically gifted students with appropriate task opportunities. Bicknell and Riley (2005) report, however, that a recent national survey found that teachers identified proportionally fewer Māori students as gifted and talented, and that this group of students is under-served in terms of culturally appropriate programmes. Without appropriate challenge, gifted students are 'at risk'; they may demonstrate boredom,

loss of interest in or commitment to mathematics, limited metacognition, and poor behaviour (Diezmann, & Watters, 1997). In accord with the policy of curriculum differentiation advocated for gifted students, (van Tassel-Baska, 1997), Diezmann and Watters (2004) suggest increasing challenge by task problematisation. Without changing the mathematical focus, a task can be problematised by methods such as inserting obstacles to the solutions, removing some information, or requiring students to use particular representations or develop generalisations. For example, the task of summing the numbers from 1 to 10 could be problematised by asking for a generalisation for the sum of the numbers from 1 to 100.

Diezmann and Watters (2002), reporting on a study involving twenty 11- to 12-year-old mathematically gifted students drawn from four mixed-ability classes, found that problematised tasks, when combined with a responsive teaching/learning environment, provided opportunities for gifted students to engage in productive mathematical activity requiring higher level cognition. "[S]tudents displayed greater persistence, collaborated with peers, demonstrated flexibility in thinking, checked work, and questioned each other" (p. 82). Studies involving younger children found that successful adaptations of tasks also included modification of games and extending manipulative use. For example, the construction of a number line to represent the distance of the ten brightest stars required the application of knowledge of large numbers, relative magnitude, and scale—and extension of the more typical use of the number line to represent sets of numbers (Diezmann & English, 2001a). According to Diezmann and Watters, the advantage of using these lateral strategies, as opposed to add-on or extension tasks, is that problematising, adapting, and enriching regular curriculum tasks provides underachieving gifted students with the opportunity to oscillate between regular activities and more challenging activities according to their capability, confidence, and motivation.

The provision of mathematical challenge is integrally linked to productive learning communities—the level of challenge affects students' involvement in the task, both in terms of opportunities for mathematical reasoning and in levels of engagement. In a comparative study of two teachers, Groves and Doig (2004) explored features of the teachers' actions. The first taught a year 1 class in Japan and the students were working on addition. The other taught an Australian year 7 class, where the students were investigating the area of a triangle. Grove and Doig explain how both teachers were able to transform familiar mathematics into challenging and enriching experiences for their students. In the teachers' views, mathematical lessons were akin to dramas that have a climactic ending. To enact the dramatic interplay, the teachers asked thought-provoking questions, they lifted the level of mathematical discussion through their discursive skills, and they pressed for understanding.

Mathematical tasks need to provide opportunities for cognitive engagement and press for understanding

> The basic aim of a mathematics lesson is for learners to learn something about a particular topic. To do this, they engage in a task ... the purpose of a task is to initiate activity by learners. (Mason & Johnston-Wilder, 2004)

"To facilitate long-term learning students need curriculum-appropriate opportunities to *develop new understandings* and to *practise and apply* their new learning" (Alton-Lee, 2003, p. 61). The development of mathematical understanding requires that learners have the opportunity and space to do 'appropriate things'. These 'things' have variously been referred to as mathematical practices (Rand Mathematics Study Panel, 2003), mathematical processes (Ministry of Education, 1992), and mathematical thinking and reasoning (Fraivillig, Murphy, & Fuson, 1999). They include: "sort, classify, structure, abstract, generalise, specialise, represent and interpret symbolically and graphically, justify and prove, encode and decode, formulate, communicate, compare, relate, recognise familiar structures, apply and evaluate applications, and automatise" (Watson, 2004, p. 364).

Classroom tasks and activities provide the vehicle for students' cognitive engagement, the

quality and level of which has a profound effect on mathematics learning outcomes (Helme & Clarke, 2001). Student cognitive engagement, qualitatively different from time on task or student participation, is influenced by the nature and implementation of available tasks. Research conducted in the QUASAR project indicates that when teachers choose tasks that require a high level of cognitive demand and set them up and implement them in ways that maintain a high level of cognitive demand, the result is an increase in student understanding and reasoning (Stein & Lane, 1996).

Stein and colleagues (1996) explored how different task demands, high or low, are placed on students. Lower level approaches to a task include memorising or reproducing learned facts, rules, formulae, and definitions, and using standard procedures or algorithms. Tasks that present high-level demands also use procedures, but in ways that build connections to mathematical meaning or require complex and nonalgorithmic thinking. Factors that support engagement with mathematical practices in the face of complex task demands include appropriate scaffolding (Anghileri, 2002), the modelling of high-level performance by the teacher and/or capable peers, the making of conceptual connections (Kazemi & Franke, 2004), the provision of appropriate amounts of time for exploring ideas and making connections (Stein et al., 1996), the encouragement of student self-monitoring (Pape et al., 2003), a sustained press for explanation, meaning, and understandings (Fraivillig et al., 1999), and the selection of tasks that build on students' prior knowledge. Many of these factors have been discussed earlier in this and the preceding chapter, so this section presents research findings specifically related to task implementation.

When implementing tasks in the classroom, it is simplistic to consider tasks as 'set' or 'fixed' (Henningsen & Stein, 1997). Each task has a relative cognitive value for an individual, and one cannot assume that the students' interpretations of that task—the activities that they engage in—are either similar to each other's or fit with the expectations of the teacher (Askew, 2004a). Stein et al.'s (1996) study of task implementation within secondary schools found that the higher the demands that a task placed on students in the set-up phase, the less likely it was that the task would be carried out faithfully during the implementation phase.

> Over half (53%) of the tasks that were set up to require the use of procedures with meaningful connections failed to keep the connection to meaning alive during implementation. … it appears as though follow-through during the implementation phase is most difficult for those kinds of task that reformers … have identified as essential to building students' capacities to engage in the processes of mathematical thinking. (p. 476)

The factors that contributed most frequently to the lowering of task demands were (a) the challenging aspects of the task, (b) a shift in focus from understanding to correctness or completeness, and (c) inappropriate allocation of time. Other factors included the teacher relaxing accountability requirements and lack of alignment between the task and students' prior knowledge, interest, and motivation.

There are similar reports of lowered cognitive demands in UK classrooms. Watson (2002) reports that teaching mathematics to low-attaining students in secondary school "often involves simplification of the mathematics until it becomes a sequence of small smooth steps which can be easily traversed" (p. 462). Frequently the teacher will take the student through the chain of reasoning and the learner merely fills in the gaps with the arithmetical answer, or low-level recall of facts. This 'path smoothing' is unlikely to lead to sustained learning since the strategy deliberately reduces a problem to what the learner can already do—with minimal opportunity for cognitive processing. This pattern of participation further reinforces the view that if students sit and do nothing for long enough, the teacher will change the requirements so that the task can be completed with minimal effort.

New Zealand classroom research (e.g., Anthony, 1994; Walls, 2004) also documents task episodes that start out as cognitively demanding and, during implementation, become rather less demanding. Based on observations of a year 12 class, Anthony (1996a) noted that reduction

in task complexity occurred when students pressured the teacher to provide explicit procedures for completing the task or when the teacher 'took over' difficult parts of the task on the students' behalf. Cognitive load was reduced by subdividing tasks, by setting short-term learning goals, or by providing informational products. For example, teacher-supplied helps such as a list of the items in a test, keywords, tables and topic summaries, all intended to support student learning, in reality substituted for student learning. This whole-class scaffolding on the pretext that all students will benefit creates a paradoxical situation. If the teacher provides unnecessary support for students who have the ability to accomplish a task without support, the relative cognitive value of a task is reduced. Unnecessary scaffolding can inhibit rather than facilitate students' learning because "some students are able to circumvent task demands or work at tasks that are below their level of ability" (Doyle, 1983, p. 180).

In addition, Anthony (1994) noted students' repeated efforts to resist task engagement. Practices regularly employed by students to reduce the cognitive load associated with a task included copying work from others, answering questions using prompts from other students, using the answers found in the back of the textbook, or offering provisional answers (guessing) to indicate apparent engagement in the task.

Stein et al. (1996) found that a classroom focus on the completeness or accuracy of answers was another significant factor associated with decline in task demand. In a series of studies involving elementary classes, Turner and Meyer (2004) also found that the "positive aspects of challenging students' thinking often are circumvented by the reliance of students and teachers on superficial indicators of understanding, such as work completion, quick responses, or correct answers" (p. 311). Students and teachers frequently seemed willing to trade the benefits of challenge-seeking (competence, pride, efficacy, and enjoyment) for the safety of avoiding mistakes and appearing competent.

In New Zealand, Walls' (2002) longitudinal study of ten primary school students noted teachers' frequent reference to children's ability in terms of completion rates rather than mathematical understanding.

> I've inherited some problems I think from other years. Standards haven't been set and kids just don't complete work and they're not use to getting, not used to actually getting through something. Finish it off. That's something I'm very tough on. I like things to be completed. (Early year 3) (Walls, 2002, p. 205)

According to Walls, this orientation towards completion appeared to be reinforced by teachers' interactions with children:

> Ms Summers: (To Peter) You've finished! Doesn't it feel good when you've done it? (Late in Y 3)
>
> Mrs Kyle: How many finished? (Looking around at the show of hands) Most of you didn't finish. You must learn to put 'DNF'—did not finish, at the bottom. (Early in Y 4)
>
> Ms Torrance: We have some amazing speedsters who have got on their rollerblades and got their two sheets done already. (p. 206)

Collectively, these studies suggest that a classroom orientation that consistently defines task outcomes in terms of the answers rather than the thinking processes entailed in reaching the answers negatively affects the thinking processes and mathematical identities of learners.

In contrast, advocates of a problem-solving curriculum (e.g., Stacey, 2005 and also earlier discussion in this chapter) argue that teaching *through* problem solving supports high-level cognitive activity.

> Problem solving activities are prominent in the recommended task only to the extent that they support the learning of other parts of the curriculum. In this vision, open problem solving is absorbed into normal teaching, as an attitude to learning and a process underpinning achievement in the normal curriculum. The goal of the

tasks highlighting problem solving is to promote good learning of routine content and to develop useful strategic and metacognitive skills, rather than explicitly to strengthen students' ability to tackle unfamiliar problems. (Stacey, 2005, p. 349)

Complex Numbers in Carmel's class

The teacher, Carmel Schettino, reports the following 'success story' from her attempts to implement a problem-solving curriculum in her pre-calculus (upper secondary school) class.

I assigned a problem that required the students to simplify the expression $(4+3i)/(1+i)$. We had discussed using the conjugate of a complex number to rewrite reciprocals of complex numbers in $a + bi$ form, but I had not given an example using conjugate multiplication as the method of 'dividing' complex numbers. However, we had discussed the transformational ramifications of the multiplication of complex numbers being a rotation and dilation of the complex number. Stephanie and Kara went to the board with their solution, although they were sure that their method was incorrect. They had collaboratively extended those transformational ideas to division, assuming that the simplified complex number resulted from a clockwise rotation (subtracting the angles) and reduction of the radius (the new radius was the quotient of the two old ones). The rest of the class was impressed with the students' ingenuity and the use of the tools to which they had already been exposed. Catie then suggested using conjugate multiplication, and Stephanie and Kara were surprised that the two answers were the same in rectangular form.

According to the teacher, this episode illustrated the creativity that was a more frequent outcome of her new problem-solving curriculum. She noted that Stephanie and Kara had come a long way from the beginning of the year. Instead of the common retort, "I have no clue," they were now able to develop an innovative method for solving the problem.

From Schettino (2003)

Stein et al.'s, (1996) study of students' engagement in mathematical tasks identifies those factors that assist the maintenance of high-level cognitive activity. The most frequent factor (in 82% of the tasks that remained high-level) was that the task built on students' prior knowledge. The second-rated factor (71%) was the provision of an appropriate amount of time for task execution. Evidence of teacher scaffolding of mathematical activity was found in 58% of the tasks that remained at high cognitive levels.

In 64% of the tasks in Stein et al.'s (1996) study that remained high-level, a sustained press for justifications, explanations, and meaning, as evidenced by teacher questions, comments, and feedback, was a major contributing factor. This factor was frequently accompanied by the modelling of competent performance by the teacher or by a capable student—often in the format of a class presentation of a solution. Presentations modelled the use of multiple representations, meaningful exploration, and appropriate mathematical justification; often, successive presentations would illustrate multiple ways of approaching a problem.

Pressing for understanding is an important aspect of quality mathematics pedagogical practice that has been noted by many researchers (Kazemi & Franke, 2004). When a teacher "presses a student to elaborate on an idea, attempts to encourage students to make their reasoning explicit, or follows up on a student's answer or question with encouragement to think more deeply" (Morrone et al., 2004, p. 29), the teacher is getting a grip on what the student actually knows and providing an incentive for them to enrich that knowledge. Morrone and colleagues provide us with examples of effective teacher talk: (1) "So in this situation how did you come up with $^{18}/_{27}$ and $^{18}/_{30}$?" (2) "When can you add the way we're adding, using the traditional algorithm, finding the common denominator? When does that make sense? Several of you started, the first thing you did was add, and you ended up, what did you end up with, with $^{19}/_{16}$. What does that mean? When can you do that? When does it make sense to add that way?" (p. 33). The following vignette illustrates the way in which a teacher established a norm of mathematical argumentation with her class.

Argumentation

In a study undertaken in the US, Forman, Larreamendy-Joerns, Stein, and Brown (1998) report on a middle school teacher who had established a shared understanding of the importance of mathematical argumentation within class discussions. She guided students into the conventions of mathematical argumentation, namely, the examination of premises, the disagreement and the counter-arguments (Lampert, 1990) and made explicit their roles and responsibilities within that argumentation process. The teacher was able to skilfully orchestrate not only the discussion so that students were aligned with the academic content at hand but also guide them into particular ways of speaking and thinking mathematically (O'Connor & Michaels, 1996) Forman et al. report that in this classroom higher level thinking was fostered as students engaged in taking and defending a position that was in opposition to the claims of other students.

The teacher in the study facilitated learning for her diverse students. She shifted students' cognitive attention from procedural rules towards making sense of their mathematical experiences. They became less engaged in finding answers to the mathematical problems than in the reasoning and thinking which lead to those solutions. In Lave's (1996) words, "[b]ecoming more knowledgeably skilled [was] an aspect of [their] participation in social practice" (p. 157). By participating in a 'microcosm of mathematical practice' (Schoenfeld, 1992), they learned how and when to participate in a discursive exchange and the meaning of an acceptable or sophisticated mathematical explanation or justification.

Unlike in conventional classrooms in which "knowledge conflicts cannot emerge" (Skovsmose, 1993, p. 176), the classroom learning community relied on disagreement and conflict resolution for the negotiation of mathematical meaning. The students learned about making inferences, analysis and generalisation. Through her skill at connecting what is normally thought of as cognitive capacities, and the social and discursive grounds on which those capacities must be maintained, the teacher enabled students to put into practice the habits of mind as well as the speech and actions valued by the community of mathematical practitioners.

From Forman, Larreamendy-Joerns, Stein, and Brown (1998)

Improvements in the quality of student engagement in the IAMP was credited by Watson and De Geest (2005), in part, to a range of task-related strategies:

> The mathematics is not simplified; there is no sense of 'finish' in the tasks, since they are stated in ways which require extended thought; there are supporting props in place (chocolate, materials, discussion); they all involve reasoning of some kind; they all contain personal challenge. (p. 39)

Improved student concentration and participation enabled tasks to be extended. The resulting sustained work on one topic promoted students' progress and awareness of progress—and hence self-esteem—that comes from being a good learner of mathematics.

In New Zealand, the formative assessment practices associated with the recently introduced National Certificate of Educational Achievement (NCEA) have the potential to affect students' task engagement. Loretz (2002), in a study of mathematics tasks used in year 11 NCEA internal assessments, found that the tasks "had moved towards high level thinking, more authentic contexts, and required more [when compared with pre-NCEA tasks] relational responses from students" (p. ii). Proponents of standards-based assessment argue that the clarity and transparency of assessment standards will reduce student anxiety, increase intrinsic motivation and self-efficacy, and promote collaboration, metacognition, and deep learning (Gipps, 1994). In particular, quality formative feedback provides students with a clear picture of what they need to do to improve (Crooks, 1988). Rawlins (in progress), working with year 12 mathematics students in three New Zealand classes, reports that students' perceptions of teacher feedback variously affected their learning strategies.

> [Feedback] tells us exactly where we went wrong; how to change it and what to

> do next time. My maths teacher shows us exactly where we went wrong. It's very helpful. I wish all teachers would.
>
> When Ms. M has time, she will write working and the correct answer on our scripts. This is very helpful. Ticks and crosses are not very helpful at all and neither is just an 'm' on the front.
>
> Ms. B is usually very good in allowing us time to read over our papers and ask any questions we have about them. I think that we need to understand what we did wrong so that we can correct it in the future. So yeah, I always use the opportunity to ask questions.

While some students appeared to want to leave the control of learning firmly with the teacher, others expressed awareness of the value of scaffolding:

> I like the teacher to give me an example, her do one herself while I watch, and then I myself attempt a similar question and see if I came to the right answer. I would want her to clearly go through the method of doing it as she did her example. I don't like being left to do it on my own the first time it frustrated me.
>
> I like to know where I went wrong and what I have to do to fix it rather than just being given the answer.

Opportunities to engage in meaningful practice activities, where the goal is to achieve understanding with fluency, are also important for learning (Marzano, 2003; Watson & Mason, 2005). Students in the IAMP study were assisted to make progress when they were given explicit guidance about 'what' they needed to remember and supported with strategies to assist them to remember.

> Remembering from lesson to lesson provides continuity and a sense of short-term progress; working deeply on mathematics can aid longer-term memory, and memory for mathematics can aid deep progress and contribute self-esteem when students are aware of the extent of what they know. (Watson, De Geest, & Prestage, 2003, p. 25)

When focusing on memory with meaning, instructional strategies include reviewing areas of confusion by means of discussion in pairs, linking to work done in previous lessons, linking topics mathematically, reminding students about the central aspects of the topic, reviewing technical terms and definitions, linking words with visual or physical memory, creating concept maps, asking students to devise their own methods of recalling and testing recall, discussing memory strategies, and targeted repetitive practice. To achieve fluency, meaningful practice opportunities include significant variations each time, providing students with a sense of the range of possibilities in a topic.

To learn mathematics effectively, it is important that *all* students have opportunities to develop and participate in mathematical practices (Diezmann et al., 2004). However, the provision of tasks that are appropriate for maximising learning opportunities and outcomes for students with special needs is less clear. In Britain, Germain (2002) examined the provision for Paul, a four-year-old child with Down's syndrome, during numeracy time in a mainstream reception class. The study focused on how learning was organised for Paul and how additional adult support was used to facilitate the learning process. Although the study did not claim general findings, acknowledging that learners with Down's syndrome show a wide range of individual differences, the case study provided evidence that Paul was able to make a positive contribution to the classroom and participate in meaningful activities. His participation was supported by structural support in the form of visual clues, such as flash cards, number stamps, or stickers, and the appropriate use of ICT.

While Paul's story is a happy one, a recent review of Australasian research (Diezmann et al., 2004) suggests that students with difficulties in mathematics may be excluded through multiple channels, including reduced access to the curriculum and altered teaching approaches. Rietvelt

(2005), in her study of children with Down's syndrome in new entrant classrooms in New Zealand, found that their opportunities to learn were masked by 'busy work' and inappropriate tasks and feedback. Ian, a case study child, was frequently praised for obtaining the 'right' answer when observations indicated that he had not developed the thinking processes that would produce those 'right' answers. Another child, Mark, was observed trying to complete counting tasks that required more advanced skills than he possessed. Jonathon, the third case study child, was frequently inhibited from full engagement with the mathematical content of the task because of inappropriate peer interactions and classroom norms.

Mathematics teaching for diverse learners utilises tools as learning supports

Our theoretical framing of situated learning lends support to the understanding that when people develop and use knowledge, they do so through their interactions with the artefacts[9] and ideas of broader social systems.

> The use of specially designed artifacts is characteristic of any human activity, and of the activities of thinking and learning in particular. Most prominent in this latter category are semiotic tools such as language, specialized symbolic systems, and educational models. (Sfard & McClain, 2002, p. 154)

In mathematics education, artefacts offer 'thinking spaces'—they are tools that help to organise mathematical thinking (Askew, 2004a; Meyer et al., 2001). Symbolic artefacts or inscriptions[10] characteristic of mathematics include the number system, algebraic symbolism, graphs, diagrams, models, equations, notations for fractions, functions, and calculus, and so on (English, 2002). Other tools include pictorial imageries, analogies, metaphors, models (such as pizzas, chocolate bars, and tens frames), examples, stories, illustrations, textbooks, rulers, clocks, calendars, technology, and problem contexts (Presmeg, 1992). In this section, we look at the way teachers use such resources to create an abstract or concrete frame of reference through which mathematical knowledge and procedures might be introduced, exemplified, and understood. In looking at how teachers use tools to support students' learning, we consider how students use tools to reorganise their activity.

The type of artefacts that teachers make available to students affects students' mathematical reasoning and performance. In a study involving the learning of measurement, Nunes, Light, and Mason (1993) explored the extent to which students' thinking could be attributed to different artefacts. They found that the choice of tools students can access does make a difference to their achievement. Sharp and Adams (2002) report how materials such as packs of gum, candy bars, pizzas, pumpkin pies, and orange juice—developed following discussion of suitable contexts with fifth grade students—became a powerful means to help students construct personal knowledge and establish a procedure for division of fractions. Blanton and Kaput's (2005) description of the tools that supported algebraic reasoning in a third grade classroom included objects such as in/out charts for organising data and concrete or visual artefacts such as number lines, diagrams, and line graphs for building and making written and oral arguments. When used effectively to support learning, these objects became referents around which students reasoned mathematically.

In the *Quality Teaching for Diverse Students in Schooling* BES, Alton-Lee (2003) provides evidence that teachers optimise student learning opportunities by complementing language use with multiple opportunities for students to access, generate, and use non-linguistic representations. This finding is particularly important in mathematics, where the use of students' inscriptions in the form of notations and graphical, pictorial, tabular, and geometric representations abound.

Inscriptions—the act of representing and the object itself—are an essential aspect of mathematics learning and teaching. From research with young children we have evidence

that the learning of mathematical notations is a constructive process. Five-year-old Paula's interpretation of capital numbers—"Thirty-three. So thirty is a capital number of three. And that's the other way to write the three" [pointing to the 3 in the tens place]—illustrates the interaction that takes place between conventional knowledge, such as notations, and children's invented mathematical notations (Brizuela, 2004). While both the conventions and the individual's inventions play a part in the creation of socially accepted knowledge and the making sense of mathematical conventions, researchers argue that the emphasis must be placed on the importance of children's inventions (Lehrer, Schabule, Carpenter, & Penner, 2000; Peters & Jenks, 2000).

In a New Zealand study, Warner (2003) tracked the development of year 5 and 6 students' notational schemes. She found that students' emerging ways of symbolising and notating provided a vehicle for communication, representation, reflection, and argumentation. Movement towards common notational practices was encouraged through the use of modelling books,[11] thinking bubbles, thinking mats and classroom norms of communication. It was noted, however, that tensions existed between formal notations and the students' idiosyncratic notational schemes. In some cases, students' attempts to adopt formal schemes acted as a barrier to the development of their mathematical understanding. Negotiating between informal and formal notation systems is a challenge for the teacher and highlights the complexity of the learning/teaching process.

Inscriptions are not limited to representations of the number system. Telling stories with graphical tools is a core element of the statistics curriculum. "A critical component in the development of students' thinking and reasoning is transnumerative thinking; that is, changing representations of data to engender an understanding of observed phenomena" (Chick, Pfannkuch, & Watson, 2005, p. 87). In order to develop statistical literacy, Chick et al. urge that students from the middle years and up need more exposure to multivariate data sets, as opposed to univariate data. These researchers argue that students should be given opportunities to create their own representations before being introduced to conventional ones, claiming that the effectiveness of standard forms may be more apparent if the students first grapple with their own representations. The following vignette illustrates how students' utilise self-generated representations to support their statistical thinking.

Telling Stories

Year 7 and 8 students, working with a set of 16 cards, are asked to look for and show any interesting features of the data. The Data Card protocol (Watson & Callingham, 1997) cards contain the name, age, weight, weekly fast-food consumption, favourite activity, and eye colour for a fictitious young person.

The researchers (Chick et al., 2005) suggest that the provision of both categorical and numerical data allows students to consider questions involving single and multiple variables and associations among variables. For example, the students who produced figure 5.8 made an appropriate transnumerative decision to calculate the average weight of the people represented in each of the 'favourite activity' categories. This represents an important first step in informal inference, where comparison of means is central to the argument and provides evidence for a difference between groups.

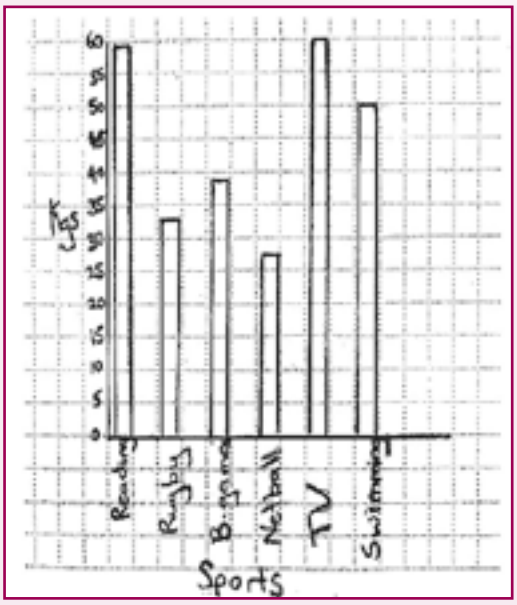
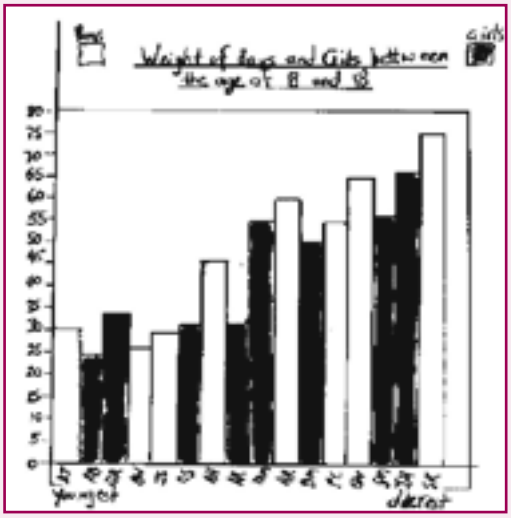


Fig. 5.8. Mean weight for each activity

When interpreting the graph in figure 5.8, the students reported surprise—they believed that the average weight for those liking swimming would be lower than for board games. This led to the development of a critical insight: "The only reason why it was high was because we didn't have enough sample ... because we only had one swimmer ... and he was quite old so he weighed quite a bit". However, despite this recognition of a third influencing variable—'age'—these students were unable to come up with a transnumerative strategy to investigate this.

According to the researchers, the capacity to transnumerate and represent three variables is important, but difficult. The students whose work is shown in figure 5.9 used transnumeration to incorporate the third variable of age.



What I've found

Analysis

From my graph I have discovered that boys usually weigh more than girls. This may be because of extra muscles and mass. Also older people are much heavier probably because of more muscles appearing as they mature.

Fig. 5.9. Weights are shown by ordering on age and highlighting gender

For these students, transnumeration of the variable 'names of students' into a new variable, 'gender', combined with the transnumeration of the variable 'age' by sorting, enabled extraction of information at a sophisticated level. The bars are used to represent weights for the ages along the horizontal axis, and colour is used to distinguish the boys from the girls. While the communication of the claim about boys weighing more than girls has not been totally successful, the increase in one numerical variable with the other is plain to see.

In each of these cases, we can see how the use of student-generated representations of the multivariate data set provided a fruitful tool to support their thinking.

From Chick et al. (2005)

Bremigan's (2005) study of 600 students' use of diagrams in an Advance Placement Calculus Examination found that the modification or construction of a diagram as part of students' problem-solving attempts was related to problem-solving success.[12] "More than 60% of students who achieved *success in set up*[13] and over 50% of students who achieved *partial success in set up* either modified or constructed diagrams" (p. 271). Moreover, the study found that errors identified in the solutions of students who achieved partial success appeared to be, at times, related to their diagrams. For instance, students often confused the direction in which a cross-section was drawn and the related choice of model (cylindrical shell or disc) when computing the volume of a solid. Diezmann and English's (2001b) research with younger children demonstrated that tools such as 'draw a diagram' are only of value when the user is sufficiently skilled to use the tool. To address their concern that primary school students often have difficulty generating effective diagrams, Diezmann (2002) trialled an instructional programme for developing students' knowledge about diagrams (e.g., networks, matrices, hierarchies, and part–whole diagrams) and their use in problem solving. They found improved problem-solving performance for all students, noting that explicit instruction about diagrams appeared to be particularly helpful for students who have difficulty identifying the structure of a problem or are easily distracted by surface details.

Research studies (e.g., Remillard & Bryans' (2004) study of TERC material in the US) have found that, within curriculum reform programmes, teachers' orientation towards using curriculum material influences the way they use it, regardless of whether they agree with the mathematical vision embedded within the material. Different uses of the material lead to different opportunities for student and teacher learning. For example, both Ell and Irwin (2006) and Higgins (2005) have found that teachers' orientation to equipment within the New Zealand NDP was an important factor in the opportunities afforded students for discussion of mathematical ideas.

Pape, Bell, and Yetkin (2003) found that one of the features of classroom instruction that emerged as critical to students' learning was the use of multiple representations. In their seventh grade classroom study, they found that, when students were engaged in solving rich problems or creating complex representations, they were motivated and accomplished significant mathematical thinking. Multiple representations lighten the cognitive load of the learner by providing conceptual tools for thinking. Anthony and Knight (1999a) reported that New Zealand teachers of students in years 4 and 5 ranked a 'think board' as the most effective resource for promoting student understanding and remembering in early computation. The 'think board' activity, which requires students to translate equations into multiple representations of stories, pictures, symbols, and real things, supports students to make connections between different representations. Lachance and Confrey (2002) also provide evidence of the value of using concrete referents that allow students to develop an understanding of mathematical symbols at the same time as they explore the connections between various types of mathematical symbol—in this case, decimals, fractions, and percentages.

Investigations of how tools support student learning point to the critical role of the teacher. Numerous researchers (e.g., Cuoco & Curio, 2001; Gravemeijer, 1997b) have found that with teacher guidance, conceptual mediating tools can act as a springboard for discussions and for structuring mathematical knowledge. In the following vignette, we can see how, in a conceptual orientation, a tool acts as an integral aspect of the learners' mathematical reasoning rather than as an external aid to it (Cobb, 2002; Gravemeijer, 1994).

Battery Power

The setting is a seventh grade teaching experiment that focused on statistics. One of the instructional goals was that the students would come to view data as a single entity rather than as a plurality of individual data values. The instructional activities involved analysing univariate data sets in order to make a decision or a judgment with the support of a computer-based analysis tool. The tool was designed to enable students to structure the data in a way that fitted with their current ways of understanding while simultaneously building toward conventional graphs.

One of the initial task situations involved comparing two separate brands of batteries, Always Ready and Tough Cell, based on a given data set of a sample of ten batteries of each of the two brands that had been tested to determine how long they would last (see fig. 5.10).

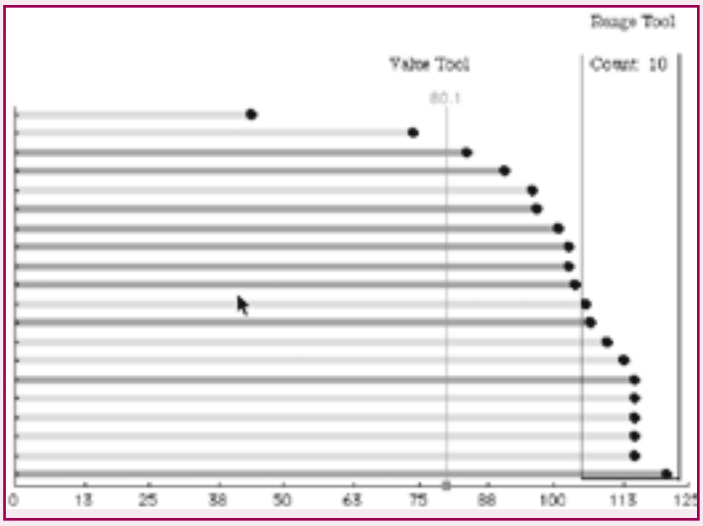


Fig. 5.10. Data set of two types of batteries

Using the computer minitool, the students were able to use the 'Value' and 'Range' functions to explore the data and make claims as to which brand was better. The ensuing justifications and meaning-making appeared to be related to the way in which individual students used the tools to support their thinking.

Celia was the first student to share her argument. She began by explaining that she used the range tool to identify the top ten batteries out of the twenty that were tested. In doing so, she found that seven of the longest lasting were Always Ready batteries. During the discussion of Celia's explanation, Bradley noted that he compared the two brands of batteries a different way.

Bradley:	Can you put the representative value of 80? Now, see there's still [Always Ready batteries] behind 80, but all the Tough Cell is above 80 and I'd rather have a consistent battery that is going to give me above 80 hours instead of one.
Teacher:	Question for Bradley? Janine?
Janine:	Why wouldn't the Always Ready battery be consistent?
Bradley:	All your Tough Cells is above 80 but you still have two behind in the Always Ready.
Janine:	Yeah, but that's only two out of ten.
Bradley:	Yeah, but they only did ten batteries and the two or three will add up. It will add up to more bad batteries and all that.
Janine:	Only wouldn't that happen with the Tough Cell batteries?
Bradley:	The Tough Cell batteries show on the chart that they are all over 80, so it seems to me they would all be better.
Janine:	[nods okay].

Bradley based his argument on the observation that all the Tough Cell batteries lasted at least 80 hours. Using the value bar to partition the data, he determined that Tough Cell was a more consistent brand. In contrast, Celia's used the range tool to isolate the ten longest lasting batteries. As the students reasoned with these tools, new meanings emerged (e.g., batteries of a particular brand are better because more of those batteries last the longest; batteries of a particular brand are consistent because all last over 80 hours).

As the discussion proceeded, Celia's choice of the 'top ten' was open to question. For instance, one student pointed out that if she had chosen the top fourteen batteries instead of the top ten, there would be seven of each brand. Celia's choice of the top ten was arbitrary in the sense that it was not grounded in the context of the investigation. Bradley, however, gave a rationale for choosing 80 hours that appeared to make sense to the students. He wanted batteries that he could be assured would last a minimum of 80 hours. As a consequence of his argument, this position was then accepted as valid.

The students continued to use the value bar to partition other data sets in this way. The researchers note that their expectation when they designed the tool was that students might use the value bar to 'eyeball' that centre of balance point of the data set. Instead, students used the value tool to partition data sets and to find the value of specific data points; they adapted the minitool to match their current ways of thinking about the data.

From McClain, Cobb, and Gravemeijer (2000)

Young-Loveridge (2005) illustrated the way in which one teacher used conceptual mediating tools to both advance students' learning and redefine the teacher–student power relationship. The investigation of the teacher's pedagogical practices took place within the context of a year 5 and 6 classroom. The class had been involved with the New Zealand NDP and the teacher had also been working on developing student-centred assessment tools. The particular tool that the teacher used for enhancing student learning took the form of 'learning logs', in which the teacher wrote a comment about an individual's learning to initiate student thinking. The learning log was then used by students to record their own learning intentions and for assembling evidence of student work that exemplified how students had met those intentions. The teacher noted that "[w]hat the learning logs have sparked for us really is the importance of the teacher–student relationship and the power that teachers have traditionally held over students, and the ways we've been breaking that down …" (p. 111).

A number of researchers (e.g., McClain & Cobb, 1999) support the usefulness of having students make written records of their work. Ball (1997) describes students' written work as an artefact of teaching and learning that provides "promise for equipping teachers with the intellectual resources likely to be helpful in navigating the uncertainties of interpreting student thinking" (p. 808). Teacher knowledge can also be enhanced: teacher development projects (e.g., Davies & Walker, 2005; NDP, 2004) provide evidence that teachers who engage in discussions about students' work and mathematical thinking develop their own mathematical awareness.

Unlike 'learning logs' as used in Young-Loveridge's study, the school mathematics text book is a familiar tool for structuring students' knowledge. Grouws, Cooney, and Jones (1988) argue that the textbook is the most important influence on students' attitudes towards mathematics. Goos (1999) vividly illustrates the way in which a year 11 teacher provided support for a class to critically engage with the ideas in a mathematics textbook. Goos observes that the readings were both assigned and spontaneous. The teacher had planned a linear process of whole-class discussion following on from the assigned reading. The students proceeded iteratively in their quest for understanding of the concepts elaborated in the reading. They read and asked questions of each other and the teacher, moving back and forth from reading to discussion in order to make sense of the reading. But it was the teacher's guidance that allowed the students to articulate in appropriate and acceptable mathematical language their conceptual understandings of the worked examples in the textbook.

In their study of students' growth in mathematical understanding, Pirie and Martin (2000)

note that a textbook or written notes may act as an important tool for students (especially at the secondary level) who are involved in what they term 'collecting'. Collecting occurs when students know what is needed to solve a problem but don't have sufficient understanding to be able to automatically recall useable knowledge. The researchers contend that the teacher can actively support students to fold back and collect by overt modelling of 'collecting' when working examples, by promotion of writing about one's understanding, by assistance with reading texts, and through student discussion and direct intervention—for example, reminding students of a particular technique in order to allow them to make progress in the building of a new concept.

Lerman (1993) records that textbooks have been shown to play a role in reflecting real-life conditions in the classroom and in reproducing existing social values. Social messages are sometimes conveyed implicitly, and the teacher who facilitates learning for diverse students is able to critically examine and challenge the assumptions on which those messages are based. Lerman points out that the most widely-used textbook series in British schools caters for different ability levels through different texts. It conveys arguable social messages. For example, while the textbook that targets top-streamed students asks them to calculate tax on an income of £50,000, the version that targets lower-ability students asks them to calculate tax on an income of £9,000. The implicit assumption is that low mathematical ability will predestine a student to a low-income future.

In the New Zealand context, Higgins et al. (2005) found texts to be an important influence in structuring teachers' own knowledge of numeracy. The teacher of an English-medium class comprising mainly Māori students used the teachers' manual (available to teachers involved in the NDP) to guide the way in which she shaped learning in the classroom. The supporting framework of the manual, combined with the pedagogical method advocated in the NDP, allowed this teacher to be more responsive to her students.

Like the teachers' manual, the concrete equipment used in the NDP plays a part in structuring knowledge. Higgins (2005) found that equipment allowed students to experience mathematical operations and relations at first hand. Through the mediating role played by equipment, new concepts were introduced, images evoked, and discussion initiated before the concepts were completely understood. However, in an earlier study, Higgins (2001) found that some teachers believed that the use of number equipment was more suitable for children in lower academic groupings. The use of equipment was also strongly linked to the notion that some students are kinaesthetic learners—a label frequently attached to Māori and Pasifika students. In the *Quality Teaching* BES, Alton-Lee (2003) contends that differential expectations for, and treatment of, Māori and Pasifika students on the grounds of supposed 'learning styles' can have negative outcomes for some students.

Teachers who foster students' mathematical development make continual inferences about the way their students 'see' the mathematical concepts embodied in the artefacts used. Reliable inferences, however, can only be made from appropriate external representational choices (English & Goldin, 2002). Ball (1993) and Lampert (1989) have found that effective teachers select and construct artefacts that their students can relate to and have the intellectual resources to make sense of and then extend their students' capacity to reason with and about the ideas under scrutiny. Walls (2004) looked at the specialised equipment, such as number lines, number tracks, and flip boards, utilised in the New Zealand Numeracy Project. She questions whether these adult-contrived artefacts are as effective for students as the familiar objects used in everyday discourse. She contrasts the number equipment with that used within The Realistic Mathematics Education (RME) Programme of the Netherlands. In his work in the RME, Gravemeijer (e.g., 1997a) has shown that through a process of generalising and formalising, meaningful equipment gradually takes on the form of its own and contributes to the shaping of mathematical reasoning. Walls makes the suggestion that New Zealand classrooms might use real life examples such as "maramataka [traditional Māori calendar], thermometers, elevator floor tracking systems, clocks, microwave timers, game boards, speedometers, and navigational

equipment" (p. 31). These conceptual tools allow students to develop facility with multiple and directional counting and "flexibility in understanding numbers sequenced in tracks that are presented in a wide range of layouts from right to left, left to right, top to bottom, bottom to top, etc." (p. 31).

Baxter, Woodward, and Olson's (2001) study in five elementary schools in reform mathematics programmes within the US noted differences among the classes in terms of the mathematical role that manipulatives played. Observations of 16 low-achieving target students revealed that whilst in some classes manipulatives were a distracter, in others they provided a conceptual scaffold. In three of the five classrooms, manipulatives became the focus rather than a means for thinking about mathematical ideas. For example, when working in pairs with fraction bars, Ginger, a target student, kept the materials in neat piles. Her partner worked on the mathematical task and simply asked Ginger for particular bars. Although the fraction bars served to involve Ginger in the task they did not appear to further her understanding of relationships among fractions. In contrast, a distinctive feature of instruction for those teachers who engaged target students in mathematical thinking was the way they used a variety of representations of a concept prior to the use of the manipulative specified in the curriculum. For example, in a geometry lesson, parallel lines were represented by a range of arm movements, lengths of string were used to create angles, calculators were used, and finally representations were transferred to geoboards. All students, including those easily distracted, worked with a wide array of geometric terms, building conceptual understandings of important mathematical ideas, such as 'parallel', rather than memorising a list of definitions generated by the teacher.

Research has shown that tools can provide effective compensatory support for students with learning disabilities. Jones et al.'s study (1996, cited in Thornton et al., 1997) illustrates how a nine-year-old learning-disabled student, Jana, used the 100s chart to move beyond pencil-and-paper computation. The episode took place within a series of lessons structured around calculations relating to 'garage sale' purchases.

> Jana spoke for herself and her partner: "We picked the picture frame for 38c and the poster for 15c—and we just have 7c change." When asked to explain how they know they would have 7c change, Jana said, "We just thought about the 100s Chart. We started with 38 and went down to 48 and then counted 5 more. So we paid 53c—that gives us 7c back because we had 60c to spend.

During early instruction with this graphic aid, Jana was encouraged to move a finger along the chart as she counted. With practice, she developed the skills to visualise the counting-on process just by thinking of the 100s chart—using the chart as a compensatory tool to compute two-digit sums mentally.

Recent research has focused on the students' use of models. Models, such as drawings, diagrams, stories, or formal–symbolic representations, can be seen as a tool for bridging the gap from concrete situations to abstract mathematics—thus acting as "a tool for strategic thinking" (van Dijk, van Oers, Terwel, & van den Eeden, 2003). However, the meanings of models developed by teachers are not always readily apparent to students—there is a danger that a ready-supplied model attempts to transmit an adult's way of thinking. The following vignette from van Dijk et al.'s experimental research study with 10 classes of grade 5 students considers how best to support students' mathematical structuring activity.

Coffee Pots

The curriculum involved open, complex problems[14] developed for percentages and graphs. The intervention consisted of 13 lessons: one lesson in which students learned about strategy use, models and their functions, and 12 lessons on percentages and graphs. In the experimental condition students co-constructively were encouraged to design their own models, as a tool for the learning of percentages and graphs. In the control condition students learned to apply ready-made models provided by the teacher. They did not learn to design or choose models themselves. For example, in the experimental condition students were asked to invent representations (designing condition) of given percentages, whereas in the control condition the students were asked to recognise percentages and shade this amount on a given model (provided condition). In both cases there was a recognition that percentages and fractions are conceptually interwoven, meaning that there was explicit emphasis on the connections between fractions and percentages.

In the following abbreviated excerpt from an experimental class, several models were presented on the blackboard for discussion. Each of the models was meant to represent a coffeepot (that holds 80 cups), which was 25% filled with coffee.

Bart:	I drew a coffeepot.
Teacher:	Ok, and how can we see it is for 25% filled with coffee?
[Some further discussion about 25%, 50% and 75% reference points.]	
Ann:	I made a sort of thermometer, and I put 100 small lines in it. And then I coloured the first 25 lines red.
Teacher:	OK, and how did you figure out how many cups of coffee this pot contains?
Ann:	I knew that 25 out of 100 is a quarter.
Teacher:	Can you show it on your model? [Anne points at her model.]
Anne:	Well, I knew that the total pot contained 80 cups, and then I took a quarter out of 80.
Teacher:	Good. Did it take you long to draw this model?
Anne:	Uh ... yes, quite long ...
Teacher:	Who can think of a solution to make this model easier to draw?
Jesse:	I would not draw all those 100 lines, but only the most important ones, like 0% and 100%.
Teacher:	OK, that would make it much easier. Linda, can you explain your model?
Linda:	I just made a kind of a bar. That's supposed to be the coffeepot. And then I shaded the first quarter of it.
Teacher:	Why did you choose a bar to represent the coffeepot? It doesn't look like a coffeepot at all!
Linda:	Because the model doesn't really have to look like a coffeepot. I think a bar like this is just easy to draw. And you can use it in every situation.
Teacher:	Well we saw three models ... What model do you like best? ... Why?
Jim:	Linda's, because it's fast, you don't need to draw the whole coffeepot. And you don't need to draw 100 lines, like in Ann's models.

The teacher then summarises the advantages of each of the models.

In contrast to the control group, in which the discussion revolved around one given model, the designing group offered various models. The invitation to explain their model and their thinking to other peers provided opportunities for others to suggest improvements and the teacher to ask critical questions to stimulate reflection to provoke improvements. The group as a whole learned about the process of model designing, and how to use models to solve math problems.

The study concluded that the strategic learning of 'how to model mathematical problems' was

beneficial to student problem solving performance. Students who learned to construct models in the experimental programme scored significantly better on the post-test than students who learned to work with models provided by the teacher. For the post-test, the effect size was .4. Although the researchers admit that no decisive, quantitative proof is given, they make a strong case that there is convincing evidence from qualitative data to support their belief that "designing models in co-construction may lead to students developing deeper insight into the meaning and use of models and consequently make possible a more flexible approach in problems solving" (p. 184).

From Van Dijk et al. (2003)

Concrete mediating tools both empower and limit mathematical thinking. Research has shown that even when tools maintain a high degree of fidelity to the problem situation, no representational context is perfect. A particular representation "may be skewed toward one meaning of a mathematical idea, obscuring other, equally important ones" (Ball, 1993, p. 162).

> Inscriptions do not merely copy the world; they select and enhance aspects of it, making visible new features and relations that cannot be seen by observing the objects and events themselves. For example, a road map selects and enhances aspects such as distance relationships and scale that are not visible to an individual at 'ground level,' while leaving out other features that are not important to the purposes of the inscription—such as trees, power lines, and buildings. (Lehrer, Schauble, & Petrosino, 2001, cited in Sfard & McClain, 2002, p, 155)

In considering the use of concrete artefacts, Askew and Millett (in press) note the possibility that students and teachers will give them different interpretations. "Being aware of the strengths and weaknesses of various artifacts is linked to subject knowledge. Simply convincing teachers that certain artifacts—number lines, counting sticks—are good things to use in lessons is not sufficient." Whereas teachers might 'see' the mathematical concept embodied in the representations, the student may see nothing more than the concrete material.

As noted earlier, classroom representational contexts "should be real or at least imaginable; be varied; relate to real problems to solve; be sensitive to cultural, gender and racial norms and not exclude any group of students; and allow the making of models" (Sullivan, Zevenbergen, & Mousley, 2002). A number of researchers provide evidence that, despite assumptions that meanings are shared, conceptual tools may actually impose different conceptualisations of apparently purely abstract mathematical problems (e.g., Masingila, 1993; Scribner, 1984). Tools or symbol systems may structure the learning system but can have different meanings for different cultural groups; the same artefact may be understood differently by different cultures (Nunes, 1997).

Authentic situations make it possible for artefacts to provide a bridge between the mathematics and the situation. Lowrie (2004) found that children involved in a planning activity (costing and scheduling a family excursion to a theme park) were assisted to 'make sense' of the task through the use of brochures, menus, bus timetables, and photographs. Students were observed to extend, adapt, and revise mathematical ideas within a personalised context. Often the problem modification was strongly influenced by personal experiences unlikely to be considered in typical problem-solving activities in this classroom. Students established their own sense of authenticity by aligning the problem with their personal experiences and understandings: "They have built more rides since I was there." [Sue]; "I didn't like that you had to have 10 minutes between each ride … because the last time I was there I only took a few minutes to walk from one ride to another" [Stuart]. The cultural artefacts provided an opportunity for all of the participants to engage in a problem scenario that was contextually meaningful. Significantly, some of the children who were not considered 'mathematically capable' invented more powerful ideas than those who did not see the task as an open-ended challenge.

As noted in earlier discussion on contextual tasks, it is a challenge to know which situations and related artefacts will appeal to or motivate learners. For some, the type of contextual material matters. Mack (1993) offered a comment from a student in her fraction study: "I don't

like pizza so I never eat it. I love ice cream and eat it everyday, so if you make the problems ice cream, it'll be easier for me." (p. 99)

As well as supporting thinking, tools provide an effective way for students to communicate their thinking. For example, Hatano and Inagaki (1998) describe an instructional episode involving first grade children in a Japanese classroom. Students familiar with join–separate problems were presented with the problem: *There are 12 boys and 8 girls. How many more boys than girls are there?* Most of the children answered correctly that there were four more boys, but one child insisted that subtraction could not be used because it was impossible to subtract girls from boys. None of the students who had answered correctly was able to argue persuasively against this assertion. It was only after the students physically modelled the situation that they realised that finding the difference was a matter of subtracting the 8 boys who could hold hands with girls from the 12 boys.

Nunes and Moreno (2002) describe a successful intervention programme for deaf children in London. The programme consisted exclusively of activities involving mathematical reasoning—as opposed to more traditional programmes consisting of teaching of algorithms. Particular emphasis was placed on visual means—such as tables and graphs—of representing relations between variables. However, because it was difficult to identify the crucial cognitive lever in this project, the researchers recommend further research to establish whether the use of visual means of communication alone can produce such positive results. They suggest that such approaches may well benefit many hearing children also.

Technological tools

How can technology be used as a resource to develop mathematical knowledge? Technological tools, like other conceptual mediators, can act as catalysts for classroom collaboration, independent enquiry, shared knowledge, and mathematical engagement. Reporting on a rich multi-dimensional task on the topic of Energy, which incorporated the use of ICT tools, Yelland (2005, p. 237) noted that opportunities for mathematical exploration included:

- looking at chronological order in sequences of *time;*
- incorporating *time* and *measurement* concepts into the editing feature of *i-movies;*
- science experiments involving *capacity*, designed to explore water consumption and conservation;
- *counting, ordering, and using number operations* for organising information for presentation to the group;
- investigations involving *money* in order to discover more effective ways of using energy such as electricity and gas;
- discussions of *space and shape* when considering, for example, storage of petroleum, transmission of electricity, and the structure and shape of electricity pylons.

Yelland notes that the children articulated their enjoyment of this project work in a variety of ways: they liked working with their friends, choosing what to do, and using computers to 'find out stuff' on the Internet and to make (PowerPoint) presentations and movies.

In the secondary environment, Arnold (2004) found that algebraic tools available on a computer not only offer mathematical insight but also make students' tacit mathematical understanding public. Likewise, Goos, Galbraith, Renshaw, and Geiger (2000) provide evidence that the graphics calculator can be a catalyst for personal (small-group) and public (whole-class) knowledge production. The calculator operates as conceptual mediator in these ways:

- message stick—students pass their calculators back and forth to compare their working and solutions;
- scratch pad—students make changes to both their own or others' work on calculators;

- partner and collaborator—students speak directly to their calculators as collaborating partners, articulating the key functions and menu choices.

The research of Goos and colleagues (2000) provides an example of how the intellectual resources of the teacher, taken together with the mediating properties of spreadsheet, function-plotting software, and graphics calculator, can contribute to the production of meaningful student knowledge about mathematical iteration. In his discursive interactions, the teacher focused on mathematical thinking. In particular, his support centred on suggestions that the students might explore both the spreadsheet and the graph, the issuing of specific challenges, suggestions that they consult with others in the class, and invitations to open their findings to class scrutiny.

In the New Zealand context, Thomas and colleagues have conducted several classroom investigations into the use of graphics calculators by secondary students. In a study that set out to compare the teaching of variables in algebra with and without the use of graphics calculators, Graham and Thomas (2001) found that students in the intervention group exhibited improved understanding of the use of letters for specific unknowns or generalised numbers. In discussing the value of the graphics calculator as a tool, Graham and Thomas argue that it:

> helps to build a versatile view of letters because the physical experience of the students is that of tapping keys to place different numbers into the calculator stores, changing the values in those stores and retrieving values from them. In this way they personally experience that letters can represent numbers; that they can have more than one value, albeit at different times; and that their value can be combined with other numbers of the value of other letters. Hence they build the idea of letter, not as a concrete object, but as a placeholder for a number or a range of numbers; as a procept. (p. 278)

In a study involving year 11 students, Hong and Thomas (2001) found that use of CAS (computer algebraic systems) calculators helped students make connections between sub-concepts across representations in a calculus unit involving the application of the Newton-Rhapson method. The researchers claim that a graphics-oriented approach to the learning of calculus can facilitate more challenging problems, with less focus on procedural skills, formulas, routine problems, and exercises. While remaining convinced of the value of graphics calculators to help students develop deeper understanding of functions, Thomas and colleagues note that research is needed to find a suitable pedagogical format and to ascertain the extent of any affective or social components to the advances in learning demonstrated in these studies.

Technological tools can serve to increase the relevance and accessibility of mathematical practices for learners (Goos & Cretchley, 2004). Nason, Woodruff, and Lesh (2002) report on a study of a sixth grade classroom, in which groups of students developed spreadsheet models to record quality of life in a number of Canadian cities. In an iterative process that began with group members' agreements on criteria for ranking cities, followed by group presentations to the class and visual investigations into each others' spreadsheets, the students produced models that allowed them to rank the cities. As part of their study, the researchers explored the potential of the computer to stimulate collaborative student efforts. As a result of public and critical scrutiny of their ideas, the students learned about mathematical efficiency and organising information for presentation. The computer became a mediator not only for building personal knowledge but also for the development of learning at the interpersonal level. It did this by occasioning interactions within and between student groups in the classroom.

Engaging with technologies can change or challenge traditional student learning trajectories. Herbert and Pierce (2005) used computer-generated modelling to challenge the traditional approach to teaching rate by studying the gradients of linear functions. They argue that the constant rate of change of linear functions does not prepare students for the complexity found in situations where the rate varies.

Rate of Change

Herbert and Pierce (2005) investigated year 10 students' use of MathWorlds[15] to explore multiple representations of the simulated movement of a lift in a multi-storey building. The experientially real context for students provided a bridge between their informal and formal knowledge, leading to a concept image of rate of change which is mathematically correct and potentially useful for further study in calculus. The movement of the lift is represented by a position-time graph, a velocity-time graph, a numeric display, a table and an algebraic rule. The interconnectivity of control between the multiple representations of the movement of the lift enables real-time investigations. For example, changing the symbolic representation of the situation results in corresponding changes in the animation, graphs and tables. Moreover, the movement of the lift is controlled by its representations and the representations of its movement are controlled by the animation.

Pre- and post-tests and interviews provided clear evidence of students' development of contextually appropriate rate-related reasoning. Post-test interviews suggested that students' concept image of rate of change was enhanced with rate-related reasoning rather than formulaic calculations of gradients used in problem solutions. For example, in the following responses it is evident that the students were thinking in terms of change in one variable per unit change in the other rather than calculating a gradient with the use of a formula.

Student A: ... plus 3 each time for the number of dollars for each bag.

Student B: It goes up by 3 for every bag bought.

Student C: ... the graph will go up $10 for each hour that he's worked, so say for 1 hour it would be up here at 30—it jumped up to 30 in the first hour and the second hour it would go to 40.

The researchers claim that the use of the motion context of lifts in MathsWorlds to develop numerous 'models of' rate of change facilitated the development of an emergent model for rate of change in other contexts. In this way velocity has become a 'model for' rate of change to solve problems in non-motion contexts thus expanding students' concept image of rate of change.

From Herbert and Pierce (2005)

Dynamic software, with its provision of a rich visual environment, can also act as a catalyst for more abstract geometric reasoning. In a year 8 Australian classroom, Vincent (2003) found that where students worked in pairs on an exploratory task using the dynamic software Cabri Geometry, there were improvements in students' arguments and their ability to connect conjecturing with proving.

Spreadsheets, an everyday software application, have also proved to be a useful mathematical tool. Calder's (2005) analysis of children's investigations illustrates how the use of spreadsheets shaped a number pattern investigation. The availability of the spreadsheet led to the children familiarising themselves with and then framing the problem through a visual, tabular lens. For example, when investigating the pattern formed by the 101 times tables, attention was focused on the discrete visual elements of the pattern rather than the consequence of an operation. However, Calder noted that while this particular medium opened up unique avenues for exploration, it simultaneously fashioned the investigation in a way that may have constrained the understanding of some children.

New technologies are not limited to computers and calculators; increasing numbers of students have access to a range of digital and mobile technologies. Davies and Hyun (2005) studied a group of 18 culturally and linguistically diverse children (aged five and six) and mapped their development of spatial representation through map-making activities. Using digital cameras, video imaging, and sketchbook drawings, the children captured real images of the area under study and downloaded the images into the computer. The children demonstrated innovative and productive problem-solving approaches as they negotiated ways to present familiar spaces. Currently, several New Zealand secondary school clusters[16] are involved in classroom-based

research studies investigating the integration of data obtained from probes (heat, motion, and light sensors) with mathematical activities.

Although increased access to the Internet is changing our everyday lives, Goos and Cretchley (2004) note that this is an area where technological advances are outpacing research into their implications for education. In their review of mathematics education research in Australasia for the period 2000–03, they conclude that "there is an ongoing need to investigate how technology-mediated communication and course delivery might lead to different kinds of teaching–learning interactions or add value to students' learning" (p. 168).

While new technologies provide an opportunity for pursuing purposeful mathematical tasks, research has found that it is their teachers' pedagogical practice that influences the way in which technological tools are used by students. Ball and Stacey (2005) suggest that teachers should share decision making about mental, pencil-and-paper, and technology-based approaches with their classes and have the students monitor their own underuse or overuse of technology. Particularly for secondary students, the use of CAS technology needs to be accompanied by the development of algebraic insight, including the ability to identify the structure and key features of expressions and to link representations. When students see an algebraic expression, they should think about what they already know about the symbols used, the structure and key features of the expression, and possibly its graph before they move further into the question. Pierce and Stacey (2004) have found that when teachers routinely demonstrate this initial step in class, it is likely to become a habit for their students.

Assessment practices, especially those involving examination questions related to the nature of algebraic and calculus procedures, are also affected with the introduction of CAS. In reviewing assessment practices in calculus examinations, Forster and Mueller (2002) note that the presence of graphics calculators requires increased awareness on the part of the teacher of the balance that needs to be maintained between opportunities for visual, empirical, and analytic approaches when solving tasks.

Pedagogical practices associated with access to resources will be increasingly affected as teachers move to incorporate technology-based presentations and web-based facilitation of learning (Heid, 2005). McHardy (2006) reports on an action research study investigating the utilisation of PowerPoint presentations and on an email discussion group with NCEA level 3 Mathematics with Calculus and Mathematics with Statistics classes. Students reported that the email contact and availability of online resources supported their learning by increasing their access to information and by giving them a more flexible work environment and greater opportunities to practise. This study provides a first glimpse at the potential and possible impacts of greater integration of technology and pedagogical practices in New Zealand mathematics classes.

Awareness of students' attitudes towards technology continues to be a central concern in evaluating the impact of technologies on mathematics learning. From recent surveys with Australian secondary students, we find mixed views. Reporting on one senior class group, Geiger (2003) notes that the students remained equivocal about the extent to which using technology to solve problems helped them understand the underlying mathematics. Reporting on two junior secondary classes, Vale (2003) found that boys reported more positive attitudes than girls towards learning mathematics with computers. These findings match the gendered patterns of behaviours that were observed when the students worked on computer-based mathematical tasks. In a much larger study, Forgasz (2002) explored the beliefs of lower secondary students in 28 schools. She found that the most traditional, gender-stereotyped beliefs about computers were held by students from higher socio-economic backgrounds or those attending higher socio-economic status schools.

Increasingly, research suggests that effective pedagogical support needs to be cognisant of the affective and social aspects of student use of technologies. For example, in a study of CAS technology, Pierce, Herbert, and Giri (2004) found that where teachers continue to privilege the high value of done-by-hand algebraic manipulations, students are unlikely to become effective

users of CAS. Students in this study perceived that CAS offered insufficient advantages over a graphics calculator to warrant the time and cognitive effort required to become effective users of this new technology. The researchers found that messages conveyed by teachers' words and actions were of paramount importance in realising the potential gains of CAS:

> [U]nambiguous value should be given to the alternative solution methods afforded by CAS. In particular this means rethinking what is meant by 'showing steps' or 'showing working' and the consequences for making. New, acceptable standards need to be established and clearly communicated to students if they are to be expected to work towards developing effective use of CAS. (p. 469)

Whatever the form of tool, quality teaching requires careful consideration of the purpose of its use, how it will be valued, and whether the outcomes are justified by the learner investment required.

Conclusion

For all students in the mathematics classroom, the 'what' that they do is crucial to their learning. The 'what'—in terms of the task set by the teacher—will be mediated by a complex array of factors, some determined by the individual student's knowledge and experiences, and others by the pedagogical affordances, constraints, and participation norms of the classroom.

Teachers' pedagogical content knowledge plays a central role in their planning and organisation for mathematics instruction. Teachers' normative ways of reasoning with instructional materials necessarily encompass both the mathematical domain that is the focus of instruction and the diverse ways students might approach and solve instructional activities. When setting tasks, teachers need to ensure that they are designed to support mathematics learning in the first instance—that they are appropriate and challenging for all students. The provision of low-level tasks accompanied by low-level expectations can limit students' mathematical development. A proficiency agenda, as advocated by Watson (2002), "does not dwell simply on the positive aspects of behaviour, motivation or attitudes, although those would play a part, it would also recognize and emphasize thinking skills which students exhibit and offer opportunity for these to be used [by all students] to learn mainstream curriculum mathematical concepts" (p. 473).

The research provides evidence that tasks vary in nature and purpose, with a range of positive learning outcomes associated with problem-based tasks, modelling tasks, and mathematics context tasks. But whatever their format, effective tasks are those that afford opportunities for students to investigate mathematical structure, to generalise, and to exemplify.

The opportunity for learning also rests with what students themselves are helped to produce. The effective use of instructional activities and tasks, alongside other resources and tools, enables students' mathematical reasoning to be visible and open for reflection. Students' own ideas are resources, both for their own and others' learning: their representations stimulate others' thinking, and their explanations challenge and extend. Support for significant mathematical thinking is dependent "on teachers who can hear the mathematics in students' talk, who can shape and offer problems of an adequate size and sufficient scope, and who can steer such problems to a productive point" (Bass & Ball, p. vii).[17]

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[1] The researchers noted, however, that while a strong mathematics curriculum, more extensive professional development and teacher support, and a whole-school reform mode made an important difference, the achievement gains were "not large enough to meet current accountability expectations under NCLB" (p. 58).

[2] Currently Mulligan and colleagues are engaged in a professional development programme aimed at developing teacher's pedagogical knowledge and children's use of pattern and structure in key mathematical concepts.

[3] The researchers Blanton and Kaput take 'algebraic reasoning' to be "a process in which students generalise mathematical ideas from a set of particular instances, establish those generalizations through the discourse of argumentation, and express them in increasingly formal and age-appropriate ways" (Blanton & Kaput, 2005, p. 413).

[4] Note: Askew interprets ? x 5 as 'something multiplied by 5' not 'something times 5'.

[5] Similarly the literature discusses 'ill-structured' problems—problems that "lack a clear formulation, or a specific procedure that will guarantee a solution, and criteria for determining when a solution has been achieved" (Kilpatrick, 1987, p. 134).

[6] Difficulties would occur where the convention is to read X x Y as X groups of Y.

[7] See Sfard, A. (2005). What could be more practical than good research? Educational Studies in Mathematics, 58, 393-413.

[8] The Dutch verb zich realise-ren means to imagine.

[9] In general the terms, artefacts and tool are used somewhat interchangeably in the literature. In a special issue of The Journal of the Learning Sciences (2002, Vol. 11) tools in the mathematics classroom are regarded as 'designed artifacts'.

[10] Some researchers more commonly use the term 'representations',

[11] For further discussion of the role of 'modelling books' in NDP classrooms see Higgins, J. & Wakefield, M. (in preparation). Modelling books and students' discussion in mathematics

[12] The researchers note that the methodology used in their study meant that it was impossible to detect if or how students who did not modify or construct a diagram used other visualisation strategies, such as the creation of mental images rather than physical images, or how they may have used the given diagrams without making markings.

[13] Success in 'set up' indicated that students were able to construct an algebraic expression, equation, or integral, which, when solved or evaluated, would lead to the problem's solution.

[14] Problems were selected from the current maths methods course used in The Netherlands and in the US (Mathematics in Context) and exercises that had been developed for MILE (Multimedia Interactive Learning Environment), a project of the Freudenthal Institute.

[15] MathWorlds is an animation software base on those obtained from the SimCalc website.

[16] Numerous workshop presentations on the application of technologies were presented at the September 2005 New Zealand Association of Mathematics Teachers Biennial Conference including a report on the current Mobile technology in the Sciences (MOTIS) project by A Tideswell.

[17] Foreword in Carpenter, T., Franke, M., & Levi, L. (2003). Thinking mathematically: integrating arithmetic and algebra in elementary school. Portsmouth: Heinemann.

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Appendix 1: Locating and Assembling BES Data

Using the 'health-of-the-system' approach, we sought to examine the various factors implicated in the creation of an effective learning community. We investigated a number of measures that fell naturally from the 'what', 'why', 'how', and 'under what conditions' questions concerning pedagogical approaches that facilitate learning for all students. The task was a considerable one, involving information management, the engagement of advisory and audit groups, and the seeking of contributions from the education community in general and the mathematics education community in particular. This level of engagement ensured that the Best Evidence Synthesis would be inclusive of views from across the community.

Our initial search strategy required us to pay attention to different contexts, different communities, and multiple ways of thinking and working. With this in mind, we undertook a literature search that crossed national and international boundaries. We used a range of search engines as well as personal networks to help us find academic journals, theses, projects, and other scholarly work with a focus on mathematics in New Zealand schools and centres, and by selected authors worldwide. We searched both print indices and electronic indices, endeavouring to make our search as broad as possible within the limits of manageability. This search took into account relevant publications from the general education literature and from the literature that relates to specialist areas of education. The search covered:

- key mathematics education literature including all major mathematics education journals (e.g., *Journal for Research in Mathematics Education*, *Educational Studies in Mathematics*, *Journal of Mathematics Teacher Education*, *For the Learning of Mathematics*, *The Journal of Mathematical Behaviour*), international conference proceedings (e.g., PME, ICME), Mathematics Research Group of Australasia publications, and international handbooks of mathematics education (e.g., Bishop et al., 2003);
- relevant New Zealand-based studies, reports, and thesis databases, supported by input from the professional community and the Ministry of Education;
- education journals (e.g., *American Educational Research Journal*, *British Educational Research Journal*, *Cognition and Instruction*, *The Elementary School Journal*, *Learning and Instruction*, etc.) and New Zealand work (e.g., SAMEpapers, SET, NZJES);
- specialist journals and projects, especially those located within the wider education field (e.g., *New Zealand Research in Early Childhood Education*; *Journal of Learning Disabilities*);
- landmark international studies including TIMSS, PISA, the UK Leverhulme projects.

This search strategy led us to a large body of literature that had something to say about facilitating mathematics learning: the total number of sourced references was just over 1100. Table 1 categorises these references by source:

Table 1: Initial Database Composition by Source

Source of data	Relative frequency (n ~1100)
Mathematics education journals	24%
Mathematics education reports, books, handbooks	16%
Mathematics education conference proceedings	15%
Theses and projects	6%
General education reports, books, handbooks	10%
General education journals, reports, and proceedings	19%
Specialist journals	10%

All entries were stored and categorised using EndNote. To assist in the initial synthesis, we distinguished between 'research' and 'discussion document', and categorised entries according to (a) our 'diversity' descriptors (e.g., ethnicity, gender, socioeconomic), (b) centre/school level, and (c) country-of-origin of the data.

These categories and sub-divisions served as a useful starting point for overviewing the literature and allowed us to foreground our fundamental intent to be responsive to diversity. In addition, by classifying entries according to sector and country of origin, we gave ourselves a constant reminder of the need to be inclusive of all perspectives and interests. This inclusiveness gave us a body of literature comprising diverse frameworks and eclectic methodological and analytic approaches.

Selecting the evidence

Given the complexity of the teaching and learning process, it is not an easy matter to link specific outcomes with specific pedagogical approaches. In our first pass through the literature, we noted that studies could claim that student achievement was *influenced* by pedagogical practice much more readily than they could explain *how* that practice affected student achievement. Many studies offered detailed explanations of student outcomes yet failed to draw conclusive evidence about how those outcomes related to specific teaching practices. Others provided detailed explanations of pedagogical practice yet made unsubstantiated claims about, or provided only inferential evidence for, how those practices connected with student outcomes.

Granted, we were not looking for linear explanations. As Sfard (2005) points out, the complexity of the teaching–learning relationship "precludes the possibility of identifying clear-cut cause–effect relationships" (p. 407). What we were searching for were studies that were able "to offer a developing picture of what it looks like for a teacher's practice to cultivate student [proficiency]" (Blanton & Kaput, 2005, p. 440). We were searching for studies that offered a "detailed look at how [teachers'] actions played out in the classroom and how students were involved in this" (Blanton & Kaput, 2005, p. 435) and the sorts of mathematical proficiency that resulted. Specifically, we were seeking studies that offered not just detailed descriptions of pedagogy and outcomes but rigorous explanation for close associations between pedagogical practice and particular outcomes.

Paying attention to diverse forms of research evidence required our serious consideration of the literature relating to disparate factors from different sectors and representative of different time periods. Luke and Hogan (in press) note that what is distinctive about the approach undertaken in the New Zealand Best Evidence programme "is its willingness to consider all forms of research evidence regardless of methodological paradigms and ideological rectitude, and its concern in finding contextually effective appropriate and locally powerful examples of 'what works'… with particular populations, in particular settings, to particular educational ends" (p. 5). We have included many different kinds of evidence that take into account human volition, programme variability, cultural diversity, and multiple perspectives. Each form of evidence, characterised by its own way of looking at the world, has led to different kinds of truth claims and different ways of investigating the truth. Our pluralist stance left us free to consider the relative strengths and weaknesses of different methodological approaches.

A fundamental challenge for this BES has been to demonstrate a basis for knowledge claims. We are absolutely aware that, like data selection, assessment of evidential claims from secondary sources is a highly perspectival activity. "Even those gazing down a microscope are as capable of finding what they expect to find, or want to find, as anyone else" (Davies, 2003). In response to this challenge, studies have been reported in a way that will make the original evidence as transparent as possible. Informed by the *Guidelines for Generating a Best Evidence Synthesis Iteration 2004 ,* we included studies that:

- provided a description of the context, the sample, and the data;

- offered details about the particular pedagogy and the specific outcomes;
- connected research to relevant literature and theory;
- used methods that allow investigation of the pedagogy–outcome link;
- yielded findings that illuminated what did or did not work.

The *Guidelines for Generating a Best Evidence Synthesis Iteration* allowed us to deal not only with a diversity of research topics, approaches, and methods, but also to capture differences in the context, practices, and ways of thinking of researchers. The method employed in this BES for evaluating validity required us to look at the ways different pieces of data meshed together and to determine the plausibility, coherence, and trustworthiness of the interpretation offered.

Assessments about the quality of research depend to a large extent on the nature of the knowledge claims made and the degree of explanatory coherence between those claims and the evidence provided. What we were looking for was the explanatory power of the stated pedagogy–outcome link. When assessing the nature and strength of the causal relations between pedagogical approaches and learning outcomes, we were guided by Maxwell's (2004) categorisations of two types of explanations of causality. The first type, the *regularity view of causation*, is based on observed regularities across a number of cases. The second type, *process-oriented explanations,* sees "causality as fundamentally referring to the actual causal mechanisms and processes that are involved in particular events and situations" (p. 4). Cobb argues (2006, personal communication) that regularity explanations are particularly useful for policy makers, while process-oriented explanations are relevant to teachers, who are concerned with "the mechanism through which and the conditions under which that causal relationship holds" (Shadish, Cook, & Campbell, 2002, p. 9, cited in Maxwell, 2004, p. 4). Attending to both types of explanation of causality meant including both large-scale and single-case studies. In many instances, we have found it useful to present a single case—a learner or teacher, a classroom, or a school—in the form of a vignette to exemplify the relations between learning processes and the means by which they are supported.

Research sources in this BES report

This BES report contains approximately 660 references. Included amongst these are research reports of empirical studies, ranging from very small, single-site settings (e.g., Hunter, 2002) to large-scale longitudinal studies (e.g., Balfanz, MacIver, and Byrnes, 2006). Some of the larger studies have multiple references because they include different papers/conference proceedings/book chapters or because they embrace work authored by different researchers (e.g., the New Zealand Numeracy Development Project). In addition, the references include reports containing educational statistics and policy, theoretical writings, and commentaries and reviews on multiple research findings (e.g., van Tassel-Baska, 1997) .

The *Guidelines for Generating a Best Evidence Synthesis Iteration* point to the importance of drawing on New Zealand research in order to illuminate what works in the New Zealand context. However, despite an exhaustive search for New Zealand work, it is apparent (see chapter 8 for further discussion) that the strengths and foci of New Zealand research are not evenly distributed. In some areas—for example, early years education—there are relatively few New Zealand (or Australian) researchers working with a specific focus on mathematics education (Walshaw & Anthony, 2004). Table 2 shows the country of origin of the literature included in this BES. The numbers reflect New Zealand's relatively new positioning within the international mathematics education research community.

Table 2: Database composition according to country

Country	Relative Frequency
New Zealand	27%
Australia	17%
United Kingdom	11%
United States	49%
Other (e.g., Africa, Netherlands, Spain)	6%

Table 3 shows the proportion of the items included in the BES (both empirical studies and commentaries) that relates to each of the different sectors. Publications relating specifically to intermediate schools have been classified with the literature on primary schools.

Table 3: Database composition according to sector

Sector	Relative Frequency (n~520)
Preschool	18%
Primary school	48%
Secondary school	21%
Teacher education	13%

Synthesising the data

Our conceptual framework, outlined in chapter 2, offered a way of structuring the data. Within the community of practice frame in and beyond the classroom, we identified the following components: (a) the organisation of activities and the associated norms of participation, (b) discourse, particularly norms of mathematical argumentation, (c) the instructional tasks, and (d) the tools and resources that learners use. We began the iterative chapter-structuring process by outlining a number of key areas. These included mathematical thinking and identities, scaffolding and co-construction, tasks, activities, assessment, educational leadership, home–school/centre links, and wider school communities. Each of these served as a starting point for our exploration and was found, in the course of the investigation, to be a useful initial category for addressing questions of equity and proficiency in relation to effective mathematics teaching.

In time, we organised these categories more cohesively into groups. What we endeavoured to do was organise multiple elements, types, and levels and varying temporal conditions in line with the critical dimensions of a community of practice and the guiding principles established in chapter 2. The content of the subsequent chapters is shaped according to these dimensions and principles. Chapter 3 focuses on all three dimensions in a search for understanding of how pedagogy influences early years outcomes. Chapters 4 and 6 explore interrelationships that are centred on the joint enterprise of developing mathematical proficiency for all learners. Chapter 5 explores the role of mathematical tasks and the part that they play in enhancing students' learning.

Reminding ourselves and readers that this BES synthesis is a product of currently accessible research, we concur with Atkinson's (2000) view that "the purpose of educational research is surely not merely to provide 'answers' to the problems of the next decade or so, but to continue to inform discussion, among practitioners, researchers and policy-makers, about the nature, purpose and content of the educational enterprise" (p. 328). Rather than offering broad answers that promise much and achieve little, it is our hope that the structure we have used will foster understanding, reflection, and action concerning the characteristics of effective pedagogical approaches in mathematics.

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Appendix 2: URLs of citations

The following 22 papers/articles/chapters/books are suggested as potentially useful sources for teachers to engage more deeply with the range of issues raised in this best evidence synthesis iteration. Readers are encouraged to source and read them. Several are available online; the others can be sourced through libraries.

The full citations are hyperlinked in the online PDF. For the convenience of those using a hard copy of the text, the URLs are listed below.

Carpenter, Thomas P ; Franke, Megan L ; Jacobs, Victoria R
A longitudinal study of invention and understanding in children's multidigit addition and subtraction
http://nzcer.org.nz/BES.php?id=BES001

Clarke, Barbara ; Clarke, Doug
Mathematics teaching in Grades K-2: painting a picture of challenging supportive, and effective classrooms
http://nzcer.org.nz/BES.php?id=BES002

Cobb, Paul ; Boufi, Ada ; McClain, Kay ; Whitenack, Jor
Reflective discourse and collective reflection
http://nzcer.org.nz/BES.php?id=BES020

Empson, Susan B
Low performing students and teaching fractions for understanding: An interactions analysis
http://nzcer.org.nz/BES.php?id=BES021

Fraivillig, Judith L ; Murphy, Laren A ; Fuson, Karen C
Advancing children's mathematical thinking in everyday mathematics classrooms
http://nzcer.org.nz/BES.php?id=BES003

Gifford, Sue
A new mathematics pedagogy for the early years: in search of principles for practice
http://nzcer.org.nz/BES.php?id=BES004

Goos, Merrilyn
Learning mathematics a classroom community of inquiry
http://nzcer.org.nz/BES.php?id=BES005

Houssart, Jenny
Simplification and repetition of mathematical tasks: a recipe for success or failure?
http://nzcer.org.nz/BES.php?id=BES006

Irwin, Kathie ; Woodward, J (paper available online)
A snapshot of the discourse used in mathematics where students are mostly Pasifika (a case study in two classrooms)
http://nzcer.org.nz/BES.php?id=BES007

Kazemi, Elham ; Stipek, Deborah
Promoting conceptual thinking in four upper-elementary mathematics classrooms
http://nzcer.org.nz/BES.php?id=BES008

Latu, Viliami (paper available online)
Language factors that affect mathematics teaching and learning of Pasifika students
http://nzcer.org.nz/BES.php?id=BES009

O'Connor, Mary Catherine
"Can any fraction be turned into a decimal?" A case study of the mathematical group discussion
http://nzcer.org.nz/BES.php?id=BES010

Rietveld, Christine M.
Classroom learning experiences of mathematics by new entrant children with Down syndrome
http://nzcer.org.nz/BES.php?id=BES011

Savell, Jan ; Anthony, Glenda Joy
Crossing the home-school boundary in mathematics
http://nzcer.org.nz/BES.php?id=BES049

Sheldon, Steven B ; Epstein, Joyce L
Involvement counts: family and community partnerships and mathematics achievement
http://nzcer.org.nz/BES.php?id=BES012

Smith, Margaret Schwan Smith ; Henningsen, Marjorie A
Implementing standards-based mathematics instruction: a casebook for professional development
http://nzcer.org.nz/BES.php?id=BES013

Steinberg, Ruth M ; Empson, Susan B ; Carpenter, Thomas P
Inquiry into children's mathematical thinking as a means to teacher change
http://nzcer.org.nz/BES.php?id=BES014

Watson, Anne ; De Geest, Els
Principled teaching for deep progress: Improving mathematical learning beyond methods and material
http://nzcer.org.nz/BES.php?id=BES015

Wood, Terry (paper available online)
What does it mean to teach mathematics differently?
http://nzcer.org.nz/BES.php?id=BES016

Yackel, Erna ; Cobb, Paul
Sociomathematical norms, argumentation, and autonomy in mathematics
http://nzcer.org.nz/BES.php?id=BES017

Young-Loveridge, Jenny (paper available online)
Students views about mathematics learning: a case study of one school involved in Great Expectations Project
http://nzcer.org.nz/BES.php?id=BES018

Zevenbergen, R
The construction of a mathematical habitus: implications of ability grouping in the middle years
http://nzcer.org.nz/BES.php?id=BES019

Appendix 3: Glossary

The page reference for the first and/or most important occurrence of the term is given in brackets.

Cognitive engagement (p. 2). The state of being engaged in thinking

Community of Practice (p. 6). The complex network of relationships within which teachers teach and students learn

Connectionist teachers (p. 97). Teachers who consistently make connections between different aspects of mathematics

Decile (p. 9). In New Zealand, a 1–10 system used by the Ministry of Education to indicate the socio-economic status of the communities from which schools draw their students; low-decile schools receive a higher level of government funding

Developmental progressions (p. 47). Sequential learning pathways categorised as a series of steps

Empirical evidence (p. 24). Data that has been collected systematically for research purposes

Equity (p. 9). The principle based on the belief that social injustices should be redressed by allocating resources according to need, not power; in education, this may mean, amongst other things, the provision of different pedagogical approaches depending upon the needs of the learners

Family Math (p. 171). A US initiative designed to develop parents' skills so they can work with their children on their mathematics

Feed the Mind (p. 45). A media campaign funded by the New Zealand Ministry of Education and designed to support family involvement in children's learning

High or low press for understanding (p. 121). Differing levels of cognitive engagement demanded of students by teachers for clarification of thinking

Kahoa (p. 36). A festive necklace (Tongan)

Kōhanga reo (p. 9). Māori-medium early childhood centres

Kura kaupapa Māori (p. 10). Māori-medium schools (kura = school), based on a Māori philosophy of learning (see pp. 54–5)

Manipulatives (p. 133). Any concrete materials used by students to model mathematical relationships

Mathematical argumentation (p. 123). Presenting a case to support or refute a premise developed by mathematical thinking

Mathematical identity (p. 19). How a student sees him/herself as a learner and doer of mathematics

Metacognition (p. 38). The knowledge and processes involved in thinking about and regulating one's own thinking, which is essential for reflecting, self-monitoring, and planning

Norms of participation (p. 54). The rules, spoken or unspoken, that govern the way students behave and contribute in the classroom

Number Framework (p. 109). A model, structured in 8 stages, showing how students typically develop understanding of number and number operations (New Zealand, NDP)

Number sense (p. 98). An understanding of the relationships, patterns, and fundamental reasonableness that lie behind all mathematical operations

Numeracy (p. 28). The ability to use mathematics effectively, fluently, and with understanding in a wide variety of contexts

Numeracy Development Project (NDP) (pp. 9, 17). The central part of the New Zealand Ministry of Education's Numeracy Strategy, which has as its primary objective the raising of student achievement in numeracy through lifting teachers' professional capability

NumPA (p. 9). A structured, diagnostic interview used by teachers to place students on the early stages of the Number Framework (New Zealand, NDP)

Open-ended tasks (p. 106). Tasks that require students to engage in problem definition and formulation before beginning to think about a solution

Pasifika students (p. 9). Students whose families have come from Sāmoa, Tonga, the Cook Islands, Niue, Tokelau, Tuvalu, and some other, smaller Pacific nations

Pedagogical Content Knowledge (p. 199). In this context, knowledge about mathematics and how to teach it as well as knowledge about how to understand students' thinking about mathematics

Pedagogy (p. 5). The processes and actions by which teachers engage students in learning

Poi (p. 26). A small ball, often made of woven flax, on the end of a length of string; swung rhythmically by women when performing action songs (Māori)

QUASAR (p. 95). A programme developed to help urban students develop understanding of mathematical ideas through engagement with challenging mathematical tasks

Revoicing (p. 78). The repeating, rephrasing, or expansion of student talk in order to clarify or highlight content, extend reasoning, introduce new ideas, or move discussion in another direction

Scaffolding (p. 27). Temporary, structured support designed to move learners forward in their thinking

School—home or home—school partnership (p. 160). The deliberate nurturing of relationships between the school and the home, in the interests of better supporting student learning

Sociocultural practices (p. 19). Practices relating to the social and cultural aspects of participation in the classroom

Sociocultural theory (p. 24). The theory that learning arises out of social interaction

Socio-economic status (SES) (p. 30). Categorisation of individuals or communities, based on income, family background, and qualifications

Sociomathematical norms (pp. 61—62). Shared understandings of the processes by which students and teacher contribute to a mathematical discussion

Tasks (p. 94). Defined by Doyle (1983) as "products that students are expected to produce, the operations that students are expected to use to generate those products, and the resources available to students while they are generating the products"

Te ao Māori (p. 54). The Māori world

Te Poutama Tau (p. 59). The Numeracy Project (New Zealand) as developed for implementation in Māori-medium schools

Te Whāriki (p. 24). The New Zealand early childhood curriculum (for children aged 5 or under)

Tukutuku panels (p. 115). A Māori craft form consisting of ornamental lattice-work panels woven together with strips of flax into intricate designs

Waiata (p. 26). A song (Māori)

Whānau (p. 41). Extended family (Māori)

Wharekura (p. 9). Māori-medium secondary schools, which are based on a Māori philosophy of learning

Zone of Proximal Development (ZPD) (p. 36). Vygotsky (1986) describes the ZPD as the "distance between the actual development level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers"

Abbreviations

CGI	Cognitively Guided Instruction Project	pp. 17, 105
EAL	English as an Additional Language	p. 116
EFTPOS	Electronic Funds Transfer at Point of Sale	p. 115
EMI-4s	Enhancing the Mathematics of Four-Year-Olds	p. 28
ENRP	Early Numeracy Research Project	p. 158
EPPE	Effective Provision of Pre-school Education Project	p. 25
ERO	Education Review Office	p. 158
IAMP	Improving Attainment in Mathematics Project	pp. 18, 99
ICME	International Congress on Mathematics Education	p. 20
ICT	Information and Communication Technologies	p. 27
IEA	International Association for the Evaluation of Educational Achievement	p. 154
IMPACT	Increasing the Mathematical Power of All Children and Teachers	p. 73
MEP	Mathematics Enhancement Project	p. 60
NCEA	National Certificate of Educational Achievement	pp. 10, 66
NEMP	National Education Monitoring Project	p. 9
NNS	National Numeracy Strategy	p. 17
PISA	Program for International Student Assessment	p. 8
REPEY	Researching Effective Pedagogy in the Early Years	p. 25
RME	Realistic Mathematics Education	p. 113
TIMSS	Third International Mathematics and Science Study	p. 14
VAMP	Values and Mathematics Project	p. 58

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